Algebra Problem Set 5

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Problem 1. *Prove* $\ell(\pi s_k) < \ell(\pi) \iff \pi(k) > \pi(k+1)$.

Solution. Suppose $\pi(k) > \pi(k+1)$ and $\pi = s_{i_1}...s_{i_l}$ is a minimal expression of π in terms of the generators s_i. By the exchange property, there exists j such that $s_{i_j}s_{i_{j+1}}...s_{j_l} = s_{i_{j+1}}...s_{j_l}s_k$ so $\pi = s_{i_1}...s_{i_{j-1}}s_{i_{j+1}}...s_{i_l}s_k$ is also a minimal expression for π. It follows $\pi s_k = s_{i_1}...s_{i_{j-1}}s_{i_{j+1}}...s_{i_l}$ and $\ell(\pi s_k) \leq \ell(\pi) - 1$.

In fact we have equality. If we had an expression for πs_k that used fewer than $\ell(\pi) - 1$ simple transpositions, $\pi s_k = s_{j_1}...s_{j_m}$, then $\pi = s_{j_1}...s_{j_m}s_k$ would have fewer than $\ell(\pi)$ transpositions. I point this out because it will come in handy in the next problem.

Conversely, suppose $\pi(k) < \pi(k+1)$. Then $\pi \circ s_k(k) > \pi \circ s_k(k+1)$. From the previous part we may conclude $\ell(\pi s_k s_k) < \ell(\pi s_k)$. But $\pi s_k s_k = \pi$. There is no equality case because π is a bijection so we are done. ■

Problem 2. *Prove that* $\ell(\pi)$ *is the same as the cardinality of the set,*

$$\text{Inv}(\pi) = \{(i,j) : 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

Solution. A Quick Lemma: If $(k, k+1) \in \text{Inv}(\pi)$ then there is a bijection,

$$\begin{aligned} \text{Inv}(\pi s_k) &\to \text{Inv}(\pi) \setminus \{(k, k+1)\} \\ (i,j) &\mapsto (s_k(i), s_k(j)). \end{aligned}$$

We have $\pi(i) = \pi \circ s_k(s_k(i))$ and $\pi(j) = \pi \circ s_k(s_k(j))$. Additionally $i < j$ iff $s_k(i) < s_k(j)$ or (exclusive) $(i,j) = (k, k+1)$. So the order relations on (i,j) only change for exactly $(k, k+1)$ which is not an element of $\text{Inv}(\pi s_k)$. It follows the map is a bijection.

Preceding to the problem, the statement is clear for $\ell(\pi) = 0$. Only e has $\ell(e) = 0$ and e fixes all elements so it has no crossings.

Assume we have the equality for n and say $\ell(\pi) = n + 1$. Write $\pi = s_{i_1} ... s_{i_{n+1}}$. If we multiply on the right by $s_{i_{n+1}}$ we get a permutation with shorter length because by canceling the $s_{i_{n+1}}$'s we get an expression written with n simple transpositions. Therefore by the previous part $(i_{n+1}, 1 + i_{n+1})$ is in the set. By our inductive hypothesis the set $\mathrm{Inv}(\pi s_{i_{n+1}})$ has n elements. By our quick lemma, $\mathrm{Inv}(\pi)$ has $n + 1$ elements. ■

Problem 3. *Let G_n be the group given by the following presentation: G_n has $n - 1$ generators, $g_1, ... g_{n-1}$ and these generators satisfy the following relations:*

$$g_i^2 = e \text{ for every } 1 \leq i \leq n$$
$$g_i g_j = g_j g_i \text{ for every } i, j \text{ such that } |i - j| > 1$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for every } 1 \leq i \leq n - 2$$

- *Prove there is a unique surjective homomorphism $G_n \to S_n$ sending g_i to s_i.*

- *Let H be the subgroup of G_n generated by $g_1, ..., g_{n-2}$. Prove the following is the list of all cosets of G_n/H:*

$$H_0 = H; H_1 = g_{n-1} H_0; H_2 = g_{n-2} H_1; ...$$
$$H_{n-1} = g_1 H_{n-2}.$$

- *Prove by induction on n that $|G_n| \leq n!$. Hence $G_n \simeq S_n$.*

Solution.

- Let $\varphi : G_n \to S_n$ be a map which takes g_i to s_i extended by linearity to be defined on all of G_n. We need to show that or any two ways of writing an element $g \in G_n$, $g_{i_1} ... g_{i_n} = g_{j_1} ... g_{j_m}$ then $s_{i_1} ... s_{i_n} = s_{j_1} ... s_{j_m}$. Equality of the two products of g_i implies they are related in some way through our list of relations. But the s_i satisfy the same relations so we can see they must also be equal.

  The map is unique because it is defined on a set which generates G_n. It is surjective because the s_i generate S_n. It is a homomorphism by its definition.

- The homomorphism of the previous part sends an element in H_i to an element of S_n which sends n to $n-i$ so the cosets are all distinct.

  It is comparatively harder to see every element of G_n is in one of these cosets. It suffices to show that if we take an element in one of these cosets and multiply it by some g_i then it remains in one of the cosets. This is sufficient because $e \in H_0$ and every element of G is a finite product of g_i's. For $g \in H_0, j > i$, we have,

  $$\begin{aligned} g_j g_i g_{i+1} \ldots g_{n-1} g &= g_i g_{i+1} \ldots (g_j g_{j-1} g_j) \ldots g_{n-1} g \\ &= g_i g_{i+1} \ldots (g_{j-1} g_j g_{j-1}) \ldots g_{n-1} g \\ &= g_i g_{i+1} \ldots (g_{j-1} g_j g_{j+1}) \ldots g_{n-1} g_{j-1} g \end{aligned}$$

  which is in the same coset. If $i = j$ then the leading $g_j g_i$ cancels. If $j = i-1$ then it shifts up to the next coset. If $j < i-1$ then we can commute it to the end. This covers all the cases of coset and generator so these cosets must capture all elements of G_n.

- G_2 has one generator and the relation that it has order 2 so $|G_2| = 2$. The subgroup H_0 introduced in the previous part is isomorphic to G_{n-1} and has index n so $|G_n| = n|G_{n-1}|$ and by induction $|G_n| = n!$. Our surjective homomorphism from the first part is therefore a bijection.

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Problem 4. *Determine the conjugacy classes in S_5 and the number of elements in each class. Then determine all Sylow subgroups of S_5.*

Solution. Conjugacy classes in S_5 are determined by cycle type. For each conjugacy class the number of elements can be determined by simple counting arguments. A representative for each class and the number of elements conjugate to it are:

e	1
$(1\ 2)$	$\binom{5}{2} = 10$
$(1\ 2\ 3)$	$2\binom{5}{2} = 20$
$(1\ 2\ 3\ 4)$	$5!/4 = 30$
$(1\ 2\ 3\ 4\ 5)$	$4! = 24$
$(1\ 2)(3\ 4)$	$\frac{1}{2}\binom{5}{2}\binom{3}{2} = 15$
$(1\ 2)(3\ 4\ 5)$	$2\binom{5}{2} = 20$

The Sylow subgroups of order 5 are generated by a 5 cycle and contain 4 nontrivial elements so there are $24/4 = 6$ of them.

Similarly the Sylow subgroups of order 3 are generated by a 3 cycle and contain 2 nontrivial elements so there are $20/2 = 10$ of them.

A Sylow 2 group is determined by a 4 cycle and a transposition moving only element moved by the 4 cycle. We know this because all subgroups of order dividing 2^3 in S_5 must be contained in a Sylow subgroup. Transpositions define a subgroup of order 2 and 4 cycles define a subgroup of order 4. Because all Sylow subgroups are conjugate to each other all Sylow 2 groups must therefore contain a 4 cycle and a transposition. If the transposition moves elements other than those moved by the 4 cycle, the subgroup would contain a 5 cycle.

It is also the case that if the 4 cycle in the Sylow subgroup is given by,

$$(1\ 2\ 3\ 4)$$

then the transposition cannot be $(1\ 2)$ because $(1\ 2)(1\ 2\ 3\ 4) = (2\ 3\ 4)$, a 3 cycle. But if it contains $(1\ 3)$ then it also contains $(2\ 4)$ because $(1\ 2\ 3\ 4)^2 = (1\ 3)(2\ 4)$. So once a 4 cycle is chosen, the transpositions that are in the Sylow subgroup containing it are determined.

All that's left to do is count the number of 4 cycles in a Sylow 2 subgroup. From the above we can see a Sylow 2 group contains at least 2 transpositions. It must also contain a copy of the Klein group we saw two problem sets ago because it has order 4 and is therefore contained in a Sylow 2 subgroup. That's 6 total elements so there is only room 2. There must be at least two because a 4 cycle's inverse is also a 4 cycle. So we see there are $30/2 = 15$ Sylow 2 groups. ■