Homework - 6

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Subject :
MATH 6111 - Abstract Algebra - I

1. **Let G be a group and V be a representation of G. Prove that we have the following decomposition as representations of G:**
$\mathrm{Sym}^2(\mathbf{V}) \oplus \wedge^2(\mathbf{V}) \cong \mathbf{V} \otimes \mathbf{V}$

   Proof :

   Since V is a G-representation, we have a map ρ : G $\longrightarrow$ GL(V) which implies that we also have a map $(\rho \otimes \rho)$ G $\longrightarrow$ (V $\otimes$ V) as $(\rho \otimes \rho)(g)(u \otimes v) = \rho(g)(u) \otimes \rho(g)(v)$ for all u,v $\in$ V, associated with the G-representation (V $\otimes$ V). Now let us define the map
φ: (V $\otimes$ V) $\longrightarrow$ (V $\otimes$ V) as φ (u$\otimes$v) = (v$\otimes$u).

   φ is a G-intertwiner.

   First we note that given any (u$\otimes$v) $\in$ (V $\otimes$ V), we have φ^2(u$\otimes$v) = $\varphi(\varphi$(u$\otimes$v)) = φ(v$\otimes$u) = u$\otimes$v implying that $\varphi^2 = \mathrm{Id}_{V\otimes V}$.

   Also, $\varphi \circ (\rho \otimes \rho) \circ \varphi^{-1}$ (u$\otimes$v) = $\varphi \circ (\rho \otimes \rho) \circ \varphi$ (u$\otimes$v) = $\varphi \circ (\rho \otimes \rho)$(v$\otimes$u) = φ (ρ(v) $\otimes$ ρ(u)) = ρ(u) $\otimes$ ρ(v) = $(\rho \otimes \rho)$(u$\otimes$v) which implies that $\varphi \circ (\rho \otimes \rho) \circ \varphi^{-1} = (\rho \otimes \rho)$ i.e. $\varphi \circ (\rho \otimes \rho) = (\rho \otimes \rho)\circ \varphi$ i.e. φ is a G-intertwiner.

   We also note that since $\varphi^2 = \mathrm{Id}_{V\otimes V}$ and $\varphi \neq \mathrm{Id}_{V\otimes V}$, its eigenvalues are +1 and -1. The respective eigenspaces are:

   E_1 = { u$\otimes$v $\in$ (V$\otimes$V) : φ(u$\otimes$v) = u$\otimes$v) } = { u$\otimes$v $\in$ (V$\otimes$V): v$\otimes$u = u$\otimes$v) } = Sym^2(V) and

   E_{-1} = { u$\otimes$v $\in$ (V$\otimes$V) : φ(u$\otimes$v) = - u$\otimes$v) } = { u$\otimes$v $\in$ (V$\otimes$V): v$\otimes$u = - u$\otimes$v) } = $\wedge^2$(V).

   Since a vector space decomposes as a direct-sum of its eigenspaces, we conclude that V$\otimes$V decomposes as:

   (V$\otimes$V) $\cong$ ($E_{-1} \oplus E_1$) $\cong$ (Sym^2(V) $\oplus$ $\wedge^2$(V)) as representations because φ is a G-intertwiner.

2. **Let G be a group and V,W be two representations of G. Recall that we have a natural map φ: $\mathbf{V}^* \otimes \mathbf{W} \longrightarrow \mathbf{Hom}_{\mathbb{C}}$ (V,W). We have to show φ is a G-intertwiner.**

   Proof:

   Since V and W are representations of G, we have homomorphisms ρ_1: G $\longrightarrow$ GL(V) and ρ_2: G $\longrightarrow$ GL(W). To show that φ is an intertwiner we have to show that, for any g$\in$G,

   g. $\varphi(\zeta \otimes$ w) . g^{-1} = φ(g.($\zeta \otimes$ w)) for any $\zeta \in V^*$ and w $\in$ W.

   Now for any v$\in$V, we have g. $\varphi(\zeta \otimes$ w) . g^{-1}(v) = g. $\varphi(\zeta \otimes$ w)(g^{-1}v) = g.(ζ(g^{-1}v)w) = ζ(g^{-1}v) g.w = (g.ζ(v)) g.w = φ (gζ $\otimes$ gw) (v) = φ(g($\zeta \otimes$ w))(v) which is exactly what is required. Therefore φ is a G-intertwiner.

3. **Let V and W be two representations of a group G. We have to show that $\mathbf{Hom}_G$(V,W) = $\mathbf{Hom}_{\mathbb{C}}(\mathbf{V},\mathbf{W})^G$ i.e. the subspace of intertwiners is actually equal to the linear-maps that are fixed under G (makes sense).**

   Proof:

   Since V and W are representations we have associated maps ρ_1 : G $\longrightarrow$ GL(V) and ρ_2 : G $\longrightarrow$ GL(W). Then we know that G acts on f$\in$ $\mathrm{Hom}_{\mathbb{C}}$(V,W) as: g.f = ρ_2(g) $\circ$ f $\circ$ ρ_1(g^{-1}). Therefore,

   f$\in$ $\mathrm{Hom}_{\mathbb{C}}(V,W)^G$ $\Leftrightarrow$ ρ_2(g) $\circ$ f $\circ$ ρ_1(g^{-1}) = f $\Leftrightarrow$ ρ_2(g) $\circ$ f = f $\circ$ (ρ_1(g^{-1}))$^{-1}$ = f $\circ$ $\rho_1(g)$ $\Leftrightarrow$ f$\in$ Hom_G(V,W).

   Hence we have that Hom_G(V,W) = $\mathrm{Hom}_{\mathbb{C}}(V,W)^G$.

4. **We have to show that every finite-dimensional representation of a finite group G has a G-invariant Hermitian inner product.**

   <u>*Proof* :</u>

   Since V is a finite dimensional representation, we can obtain an orthonormal basis of V, say $\{e_i\}_{i=1}^{m}$ via Gram-Schmidt-Orthogonalization-Algorithm, where dim(V) = m.

   Then we can define a Hermitian-Inner-Product on V by defining $\langle e_i , e_j \rangle = \delta_{ij}\ \forall\ 1 \leq i,j \leq m$. This inner-product is Hermitian through basic facts of Linear Algebra.

   Now we define another Hermitian-Inner-Product (,) on V as: $(u,v) = \frac{1}{|G|} \sum_{g\in G} \langle gu,gv\rangle$, $\forall$ u,v$\in$V. This can be done because $|G|$ is finite.

   Since the inner-product $\langle , \rangle$ was Hermitian (,) will also be Hermitian – linearity extends by linearity of the $\langle\ ,\ \rangle$ and linearity of g action, non-negativity of $\langle\ ,\ \rangle$ implies that of (,). Now we have to show that (,) is G-invariant.

   Let h$\in$G and u,v$\in$V then $(hu,hv) = \frac{1}{|G|} \sum_{g\in G} \langle g(hu),g(hv)\rangle = \frac{1}{|G|} \sum_{g\in G} \langle gh(u),gh(v)\rangle = \frac{1}{|G|} \sum_{h'\in G} \langle h'u,h'v\rangle = (u,v)$. Therefore (,) is G-invariant.

5. **We have to use the problem above to (re)prove the Maschke's Theorem: Let $V_1 \subset V$ be a subrepresentation of a finite dimensional representation. Then $\exists\ V_2 \subset V$ such that $V \cong V_1 \oplus V_2$.**

   <u>*Proof* :</u>

   From the above problem, we know that V has a G-invariant Hermitian inner product (,). Given a sub-representation V_1 of V, we define $V_2 = \{u\in V : (u,v) = 0\ \forall\ v\in V_1\}$.

   Then $V_2 \subset V$ as a vector space because we know that orthogonal-complement of a subspace is another subspace, from elementary Linear Algebra.

   V_2 is G-stable.

   Let u$\in V_2$ then (u,v) = 0 $\forall$v$\in V_1$. Let g$\in$G then we see that $(g.u,v) = g^{-1}(g.u,v)$ since (,) is G-invariant. So, $(g.u,v) = g^{-1}(g.u,v) = (g^{-1}g.u,\ g^{-1}v) = (u,\ g^{-1}v = 0$ since u$\in V_2$. Therefore, $\forall$ u$\in V_2$, g.u$\in V_2$ implying that V_2 is G-stable.

   Now to show that $V \cong V_1 \oplus V_2$.

   First of all we note that if v$\in(V_1 \cap V_2)$ then (v,v) = 0 implying that v = 0 and hence $V_1 \cap V_2 = \{0\}$.

   Now since V is a finite-dimensional representation, we can use Gram-Schmidt-Orthogonalization to obtain a basis $\{e_i\}_{i=1}^{m}$. Out of these basis- vectors some will form basis of V_1 and others of V_2. Let $\{e_i\}_{i=1}^{m_1}$ be the basis of V_1 and $\{e_i\}_{i=m_1+1}^{m}$ be that of V_2. Then given any element of v$\in$ V we have that $v = \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m_1} a_i e_i + \sum_{i=m_1}^{m} a_i e_i$ where the first summand is in V_1 and second in V_2.

   $V_1 \cap V_2 = \{0\}$ in conjunction with the fact that every vector in V can be written as a sum of vectors in V_1 and V_2 implies that $V \cong V_1 \oplus V_2$ as representations because both of them are G-Stable.