Problem 1 Part 1

Recall that we have the following isomorphisms of vector spaces:

$$\mathbb{C}G \xrightarrow{\Psi} \bigoplus_{\lambda\in\Lambda_G} \mathrm{End}_{\mathbb{C}}\left(V_\lambda\right) \xleftarrow{\Phi} \bigoplus_{\lambda\in\Lambda_G} V_\lambda^* \otimes V_\lambda$$

where Λ_G is the set of isomorphism classes of finite-dimensional irreducible representations of G.

Let us be more explicit in our treatment of Ψ and Φ. If $\xi_\lambda \in V_\lambda^*$ and $w_\lambda \in V_\lambda$, then Φ is given by

$$\Phi\left(\bigoplus_{\lambda\in\Lambda_G} \xi_\lambda\left(\cdot\right) \otimes w_\lambda\right) = \bigoplus_{\lambda\in\Lambda_G} \xi_\lambda\left(\cdot\right) w_\lambda,$$

where "$(\cdot)$" is a placeholder for the λ coordinate of an element of $\oplus_{\lambda\in\Lambda_G} V_\lambda$. As for Ψ, we know that $\left(e_g\right)_{g\in G}$ is a basis for $\mathbb{C}G$ if we define $e_g\left(x\right) = \delta_{g,x}$ for all $g, x \in G$. Also, for each $\lambda \in \Lambda_G$, there is a map $\rho_\lambda : G \to GL\left(V_\lambda\right)$ corresponding to the irreducible representation of G on V_λ. With these conventions, Ψ is given by

$$\Psi\left(\sum_{g\in G} a_g e_g\left(\cdot\right)\right) = \bigoplus_{\lambda\in\Lambda_G} \sum_{g\in G} a_g \rho_\lambda\left(g\right)\left(\cdot\right)$$

where $a_x \in \mathbb{C}$ for every $x \in G$, "$(\cdot)$" is a placeholder for an element of G on the left-hand side of the equation, and "$(\cdot)$" is a placeholder for the λ coordinate of an element of V on the right-hand side of the equation.

Given $\xi \in V_\lambda^*$ and $v \in V_\lambda$, define $\varphi_{\xi,v} : G \to \mathbb{C}$ by $g \mapsto \xi\left(\rho_\lambda\left(g^{-1}\right)\left(v\right)\right)$. Prove

$$\Psi\left(\varphi_{\xi,v}\right) = \Phi\left(\left(\frac{|G|}{\dim\left(V_\lambda\right)}\right)\xi \otimes v\right).$$

<u>Proof:</u> We know that for any $\xi \in V_\lambda^*$ and $w \in V_\lambda$, $\rho_\lambda^*\left(g\right)\left(\xi\right) \otimes \rho_\lambda\left(g\right)\left(w\right) \in V_\lambda^* \otimes V_\lambda$ for all $g \in G$, and thus $\sum_{g\in G} \rho_\lambda^*\left(g\right)\left(\xi\right) \otimes \rho_\lambda\left(g\right)\left(w\right) \in V_\lambda^* \otimes V_\lambda$. Next, observe that the restriction of Φ to a particular coordinate $\lambda \in \Lambda_G$ is the natural map between $V_\lambda^* \otimes V_\lambda$ and $\mathrm{End}_{\mathbb{C}}\left(V_\lambda\right)$, which we know by problem 4 from problem set 6 is G-intertwining. Also, by Schur's lemma, every element of $\mathrm{End}_{\mathbb{C}}\left(V_\lambda\right)$ is a constant multiple of the identity map Id_{V_λ}. Therefore,

$$\Phi\left(\sum_{g\in G} \rho_\lambda^*\left(g\right)\left(\xi\right) \otimes \rho_\lambda\left(g\right)\left(w\right)\right) = z\,\mathrm{Id}_{V_\lambda}$$

for some constant z. Since Φ is an isomorphism, we know that traces are preserved under Φ. In particular, since the trace of $\rho_\lambda^*\left(g\right)\left(\xi\right) \otimes \rho_\lambda\left(g\right)\left(w\right)$ is precisely $\xi\left(w\right)$, the above equation yields $|G|\,\xi\left(w\right) = z\dim\left(V_\lambda\right)$. Evaluating both sides of this equation at any $v \in V_\lambda$ yields

$$\sum_{g\in G} \xi\left(\rho_\lambda\left(g^{-1}\right)\left(v\right)\right) \cdot \rho_\lambda\left(g\right)\left(w\right) = \left(\frac{|G|}{\dim\left(V_\lambda\right)}\right)\xi\left(w\right) \cdot v.$$

But then

$$\begin{aligned}\Psi\left(\xi\left(\rho_\lambda\left(g^{-1}\right)(v)\right)\left[\rho_\lambda(g)(w)\right]\right) &= \sum_{g\in G}\rho_\lambda\left(g^{-1}\right)(v)\,\rho_\lambda(g)(w)\\ &= \left(\frac{|G|}{\dim\left(V_\lambda\right)}\right)\xi(w)\cdot v\\ &= \Phi\left(\left(\frac{|G|}{\dim\left(V_\lambda\right)}\right)\xi(w)\otimes v\right),\end{aligned}$$

as desired.

Problem 1 Part 2

Consider the $G\times G$ action on $\mathbb{C}G$ by the following formula, for every $g_1,g_2,x\in G$ and $f\in\mathbb{C}G$:

$$\left[\left(g_1,g_2\right)\cdot f\right](x) = f\left(g_1^{-1}xg_2\right).$$

Prove that $\Phi^{-1}\circ\Psi$ is a $G\times G$-intertwiner where $G\times G$ acts on $V_\lambda^*\otimes V_\lambda$ by $\left(g_1,g_2\right)\cdot(\xi,v) = \left(g_1\cdot\xi\right)\otimes\left(g_2\cdot v\right)$.

<u>Proof:</u> It is clear that the two actions stated in the problem statement are indeed actions. To prove that $\Phi^{-1}\circ\Psi$ is $G\times G$-intertwining, using what we know from part 1, we have

$$\begin{aligned}\Phi^{-1}\circ\Psi\circ\left(g_1,g_2\right)\cdot(\xi,v) &= \Phi^{-1}\circ\Psi\left(g_1\cdot\xi\otimes g_2\cdot v\right)\\ &= \xi\left(g_1^{-1}xg_2\right)v\\ &= \left(g_1,g_2\right)\cdot\xi(x\cdot v)\\ &= \left(g_1,g_2\right)\cdot\left(\Phi^{-1}\circ\Psi(\xi,v)\right)(x),\end{aligned}$$

as desired.

Problem 1 Part 3

Define a map $*:\mathbb{C}G\times\mathbb{C}G\to\mathbb{C}G$ as follows: For every $f_1,f_2\in\mathbb{C}G$,

$$f_1 * f_2(x) = \sum_{g\in G} f_1\left(xg^{-1}\right)f_2(g).$$

Prove that $\Psi\left(f_1 * f_2\right) = \Psi\left(f_1\right)\circ\Psi\left(f_2\right)$ where the operation on the right-hand side is composition of linear endomorphisms.

<u>Proof:</u> If we write $f_1 = \sum_{g\in G} a_g e_g$ and $f_2 = \sum_{g\in G} b_g e_g$, then we can compute

$$\Psi\left(f_1 * f_2\right) = \Psi\left(\sum_{g\in G} f_1\left(xg^{-1}\right)f_2(g)\right)$$

$$\begin{aligned}
&= \Psi\left(\sum_{g\in G}\sum_{s\in G} a_s e_s\left(xg^{-1}\right)\sum_{t\in G} b_t e_t\left(g\right)\right)\\
&= \bigotimes_{\lambda\in\Lambda(G)}\sum_{g\in G} a_g\rho_\lambda\left(g\right)\circ\bigotimes_{\lambda\in\Lambda_G}\sum_{g\in G} b_g\rho_\lambda\left(g\right)\\
&= \Psi\left(\sum_{g\in G} a_g e_g\right)\circ\Psi\left(\sum_{g\in G} b_g e_g\right)\\
&= \Psi\left(f_1\right)\circ\Psi\left(f_2\right).
\end{aligned}$$

This completes the proof.

Problem 3

Consider the action of the symmetric group S_n on $V \subset \mathbb{C}^n$ by permutation of coordinates, where

$$V := \left\{\sum_{i=1}^{n} a_i e_i : a_1 + \cdots + a_n = 0\right\}$$

where $(e_i)_{i=1}^n$ is a basis for $\mathbb{C}^n$. Prove that V is an irreducible representation of S_n.

Proof: We should first confirm that S_n does indeed act on V, though this is an easy task. We already know that S_n acts on $\mathbb{C}^n$ by permuting the basis vectors e_1 through e_n. Furthermore, given any $\sigma \in S_n$ and $v = \sum_{i=1}^n a_i e_i \in V$, then $\sum_{i=1}^n a_i = 0$, and

$$\rho_\sigma v = \sum_{i=1}^{n} a_i \rho_\sigma e_i = \sum_{i=1}^{n} a_i e_{\sigma(i)} = \sum_{i=1}^{n} a_{\sigma^{-1}(i)} e_i,$$

which belongs to V since $\sum_{i=1}^n a_{\sigma^{-1}(i)} = \sum_{i=1}^n a_i = 0$. Thus V is S_n-stable. Hence V is indeed a representation of S_n.

Next, I claim that $\{e_1 - e_i\}_{i=2}^n$ is a basis for V. Certainly $e_1 - e_i \in V$ for all $2 \le i \le n$. For linear independence, suppose $\sum_{i=2}^n a_i\left(e_1 - e_i\right) = 0$ for some constants a_2 through a_n. Then $\left(\sum_{i=2}^n a_i\right) e_1 - \sum_{i=2}^n a_i e_i = 0$. Since the e_i's are independent, we know that $\sum_{i=2}^n a_i = -a_2 = \cdots = -a_n = 0$. So then $a_2 = \cdots = a_n = 0$ and $0 = \sum_{i=2}^n a_i$. Therefore, the $e_1 - e_i$'s are linearly independent. To prove that $\{e_1 - e_i\}$ spans V, let $v = \sum_{i=1}^n a_i e_i \in V$. Then

$$\begin{aligned}
v &= \sum_{i=1}^{n} a_i e_i = a_1 e_1 + \sum_{i=2}^{n} a_i e_i = a_1 e_1 + \sum_{i=2}^{n}\left(-a_i\left(e_1 - e_i\right) + a_i e_1\right)\\
&= a_1 e_1 + \left(\sum_{i=2}^{n} -a_i\left(e_1 - e_i\right)\right) + \sum_{i=2}^{n} a_i e_1 = \left(e_1\sum_{i=1}^{n} a_i\right) + \sum_{i=2}^{n} -a_i\left(e_1 - e_i\right)\\
&= \sum_{i=2}^{n} -a_i\left(e_1 - e_i\right).
\end{aligned}$$

Thus $\{e_1 - e_i\}$ spans V. Hence $\{e_1 - e_i\}$ is a basis for V.

I now claim that, by an argument similar to the one from the above paragraph, that $\{e_i - e_{i+1}\}_{i=1}^{n-1}$ is a basis for V. Certainly $e_i - e_{i+1} \in V$ for all $1 \leq i \leq n-1$. For linear independence, suppose $\sum_{i=1}^{n-1} a_i (e_i - e_{i+1}) = 0$ for some constants a_1 through a_{n-1}. Then $a_1 e_1 + \sum_{i=2}^{n-1} (a_i - a_{i-1}) e_i - a_{n-1} e_n = 0$. Since the e_i's are independent, we know that $a_1 = (a_2 - a_1) = \cdots = (a_{n-1} - a_{n-2}) = a_{n-1} = 0$. So then $a_1 = a_{n-1} = 0$, which combined with $a_2 - a_1 = a_{n-1} - a_{n-2} = 0$ yield $a_2 = a_{n-2} = 0$, and continuing this procedure yields $a_1 = \cdots = a_{n-1} = 0$. Therefore, the $e_i - e_{i+1}$'s are linearly independent. To prove that $\{e_i - e_{i+1}\}$ spans V, let $v = \sum_{i=1}^{n} a_i e_i \in V$. Then

$$\begin{aligned} v &= \sum_{i=1}^{n} a_i e_i = a_n e_n + \sum_{i=1}^{n-1} (a_i e_i + a_i e_{i+1} - a_i e_{i+1}) = a_n e_n + \sum_{i=1}^{n-1} [a_i (e_i - e_{i-1}) + a_i e_{i+1}] \\ &= \left(a_n e_n + \sum_{i=1}^{n-1} a_i e_{i+1}\right) + \sum_{i=1}^{n-1} a_i (e_i - e_{i-1}) = \sum_{i=1}^{n-1} a_i (e_i - e_{i-1}) . \end{aligned}$$

The final inequality above holds since the sum of the coefficients of $a_n e_n + \sum_{i=1}^{n-1} a_i e_{i+1}$ is precisely $\sum_{i=1}^{n} a_i e_i = 0$. Thus $\{e_i - e_{i+1}\}$ spans V. Hence $\{e_i - e_{i+1}\}$ is a basis for V.

Proceeding to the given problem, suppose W is a nonzero irreducible subrepresentation of V. We must prove that $W = V$. Let $w \in W \setminus \{0\}$, and write $w = \sum_{i=2}^{n} a_i (e_1 - e_i)$, which we know is possible by our previous paragraphs. Let $s_i = (i, i+1) \in S_n$ for $1 \leq i \leq n-1$. Assume for contradiction that $\rho_{s_i} w = w$ for all $1 \leq i \leq n-1$. So for each $2 \leq j \leq n-1$,

$$\sum_{i=2}^{n} a_i (e_1 - e_i) = w = \rho_{s_j} w = a_{j+1} (e_1 - e_j) + a_j (e_1 - e_{j+1}) + \sum_{\substack{2 \leq i \leq n, \\ i \neq j, j+1}} a_i (e_1 - e_i) .$$

Canceling terms from both sides of the above equation yields $-a_j e_j - a_{j+1} e_{j+1} = -a_{j+1} e_j - a_j e_{j+1}$, and so $(a_{j+1} - a_j) e_{j+1} = (a_{j+1} - a_j) e_j$, which implies $a_j = a_{j+1}$. Since this holds for all $2 \leq j \leq n-1$, we have $a_2 = \cdots = a_n$. Writing w as a sum of e_1 through e_n then yields

$$w = \left(\sum_{i=2}^{n} a_i\right) e_1 - a_2 e_2 - \cdots - a_n e_n = (n-1) a_2 e_1 - a_2 e_2 - \cdots - a_2 e_n.$$

It is apparent that $\{e_n - e_i\}_{i=1}^{n-1}$ is a basis for V, and the proof of this is identical to the proof of the fact that $\{e_1 - e_i\}_{i=2}^{n}$ is a basis for V up to appropriate changes in subscripts. Writing $w = \sum_{i=1}^{n-1} b_i (e_n - e_i)$ for constants b_1 through b_{n-1} and following the procedure of the above paragraph then yields $b_1 = \cdots = b_{n-1}$, and writing w as a sum of e_1 through e_n then yields

$$w = -b_1 e_1 - \cdots - b_{n-1} e_{n-1} + \left(\sum_{i=1}^{n-1} b_i\right) e_n = -b_1 e_1 - \cdots - b_1 e_{n-1} - (n-1) b_1 e_n.$$

The only way to reconcile our two expressions of w involving a_2 and b_1 is if $a_1 = b_1 = 0$. But then $a_2 = \cdots = a_n = 0$, which implies $w = 0$. This which contradicts the fact that we chose $w \in W \setminus \{0\}$. Therefore, there is some $1 \leq j \leq n-1$ such that $\rho_{s_j} w = w$.

Let $1 \leq j \leq n-1$ be such that $\rho_{s_j} w \neq w$. Then

$$w - \rho_{s_j} w = a_j (e_1 - e_j) + a_{j+1} (e_1 - e_j) - a_{j+1} (e_1 - e_j) - a_j (e_1 - e_{j+1}) = -a_j (e_j - e_{j+1}).$$

Since $w - \rho_{s_j} w \neq 0$, $a_j \neq 0$. Therefore $-a_j$ is invertible, and since $w, \rho_{s_j} w \in W$, we know that W contains $(-a_j)^{-1} \left(w - \rho_{s_j} w\right) = e_j - e_{j+1}$. Then since $\rho_{s_i} (e_j - e_{j+1}) \in W$ for all $1 \leq i \leq n-1$, W contains $e_i - e_{i+1}$ for $1 \leq i \leq n-1$. We have proven that $\{e_i - e_{i+1}\}_{i=1}^{n-1}$ is a basis for V, and thus W contains a basis for V, which means that $W = V$. Thus V is the only nonzero irreducible subrepresentation of V. Hence V is irreducible. This completes the proof.

Problem 4

Recall that D_n is the group generated by two elements s_1, s_2, subject to the relations $s_1^2 = s_2^2 = (s_1 s_2)^n = 1$. Consider the action of the dihedral group D_n on $\mathbb{C}^2$ given by the group homomorphism $D_n \to GL_2(\mathbb{C})$:

$$s_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad s_2 \mapsto \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $\theta = 2\pi/n$. Prove that for $n \geq 3$, the resulting representation is irreducible.

<u>Proof:</u> We must first check confirm that the given action of D_n on $\mathbb{C}^2$ is indeed a group homomorphism, which we will call ρ. Since we have defined ρ on a generating set for D_n, ρ is automatically defined on all of D_n so as to satisfy $\rho_{gh} = \rho_g \rho_h$ for all $g, h \in D_n$. Since the generators s_1 and s_2 for D_n are subject to the relations $s_1^2 = s_2^2 = (s_1 s_2)^n = 1$, it suffices to prove that the images of s_1 and s_2 under ρ satisfy the same relations that s_1 and s_2 satisfy, i.e. $\rho_{s_1}^2 = \rho_{s_2}^2 = (\rho_{s_1} \rho_{s_2})^n = I$ where I is the identity map on $\mathbb{C}^2$. To this end, we simply compute

$$\begin{aligned} \rho_{s_1^2} &= \rho_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \rho_{s_1} \rho_{s_1}, \\ \rho_{s_2^2} &= \rho_1 = I = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \rho_{s_2} \rho_{s_2}. \end{aligned}$$

For the final relation, first observe that

$$\rho_{s_1} \rho_{s_2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Let us to use A_θ to denote the above matrix for $\rho_{s_1} \rho_{s_2}$. Now observe that for any two angles θ and ψ,

$$A_\theta A_\psi = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi & \cos\theta\sin\psi + \sin\theta\cos\psi \\ -\sin\theta\cos\psi - \cos\theta\sin\psi & -\sin\theta\sin\psi + \cos\theta\cos\psi \end{bmatrix} \\
&= \begin{bmatrix} \begin{array}{c} (\cos(\theta+\psi)+\cos(\theta-\psi))/2 \\ -(\cos(\theta-\psi)-\cos(\theta+\psi))/2 \end{array} & \begin{array}{c} (\sin(\theta+\psi)-\sin(\theta-\psi))/2 \\ (\sin(\theta+\psi)+\sin(\theta-\psi))/2 \end{array} \\ \\ \begin{array}{c} -(\sin(\theta+\psi)+\sin(\theta-\psi))/2 \\ -(\sin(\theta+\psi)-\sin(\theta-\psi))/2 \end{array} & \begin{array}{c} -(\cos(\theta-\psi)-\cos(\theta+\psi))/2 \\ +(\cos(\theta+\psi)+\cos(\theta-\psi))/2 \end{array} \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\psi) & \sin(\theta+\psi) \\ -\sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix}.
\end{aligned}
$$

Using this fact, I claim that $(\rho_{s_1}\rho_{s_2})^m = A_{\theta m}$ for all $m \in \mathbb{N}\,(\not\ni 0)$. This is clear by induction: for $m = 1$, we've already proven that $\rho_{s_1}\rho_{s_2} = A_\theta$, and if $(\rho_{s_1}\rho_{s_2})^m = A_{\theta m}$, then by the above fact, $(\rho_{s_1}\rho_{s_2})^{m+1} = A_{\theta m}A_\theta = A_{\theta(m+1)}$. Hence $(\rho_{s_1}\rho_{s_2})^m = A_{\theta m}$ for all $m \in \mathbb{N}$. In particular,

$$(\rho_{s_1}\rho_{s_2})^n = A_{\theta n} = A_{2\pi n/n} = A_{2\pi} = \begin{bmatrix} \cos(2\pi) & \sin(2\pi) \\ -\sin(2\pi) & \cos(2\pi) \end{bmatrix} = I.$$

Thus ρ_{s_1} and ρ_{s_2} satisfy the same relations as s_1 and s_2, which confirms that ρ is well-defined. Combining this with the fact that $\rho_{gh} = \rho_g\rho_h$ for all $g, h \in D_n$ confirms that ρ is a group homomorphism. Hence D_n does indeed act on $GL_2(\mathbb{C})$ as described in the problem statement.

To prove that ρ is irreducible, assume for contradiction that ρ is not irreducible. This means that $\mathbb{C}^2$ has some G-invariant subspace $V \subset \mathbb{C}^2$. Since $\dim\mathbb{C}^2 = 2$, $\dim V = 1$. So for any $v \in V \setminus \{0\}$, $V = \operatorname{Span}(v)$, which means that for all $g \in D_n$, $\rho_g v \in V = \operatorname{Span}(v)$, so $\rho_g v = \lambda v$ for some $\lambda \in \mathbb{C}$. Thus v is an eigenvector for ρ_g for all $g \in D_n$. One one hand, this means that v is an eigenvector for ρ_{s_1}. Since ρ_{s_1} is a diagonal matrix, we can easily see that it has eigenvalues 1 and -1. Since we have

$$
\begin{aligned}
\rho_{s_1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\rho_{s_1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned}
$$

we see that 1 and -1 have corresponding eigenvectors $(1,0)$ and $(0,1)$. Since $V = \operatorname{Span}(v)$, we can conclude that $V = \mathbb{C}\times\{0\}$ or $V = \{0\}\times\mathbb{C}$. On the other hand, the fact that v is an eigenvector for ρ_g for all $g \in D_n$ implies that $\rho_{s_2}v = \lambda v$ for some $\lambda \in \mathbb{C}$. Writing $v = (v_1, v_2)$, this means

$$\begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}\right) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\cos\left(\frac{2\pi}{n}\right) + v_2\sin\left(\frac{2\pi}{n}\right) \\ v_1\sin\left(\frac{2\pi}{n}\right) - v_2\cos\left(\frac{2\pi}{n}\right) \end{bmatrix} = \lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Since $V = \mathbb{C}\times\{0\}$ or $V = \{0\}\times\mathbb{C}$, either $v_1 = 0$ or $v_2 = 0$. If $v_1 = 0$, then $v_1\cos(2\pi/n) + v_2\sin(2\pi/n) = \lambda v_1$ becomes $v_2\sin(2\pi/n) = 0$. Since $n \geq 3$, this implies $v_2 = 0$, and so $v = 0$, which contradicts that $v \in V\setminus\{0\}$. If instead $v_2 = 0$, then $v_1\sin(2\pi/2) - v_2\cos(2\pi/n) = \lambda v_2$ becomes $v_1\sin(2\pi/2) = 0$, which implies $v_1 = 0$, and so $v = 0$, which contradicts that $v \in V\setminus\{0\}$. We have thus arrived at contradictions in all scenarios. Therefore, it cannot be the case that ρ is not irreducible. Hence ρ is irreducible.

Problem 6

Let G be a group action on a finite nonempty set X. Let $\mathscr{F}(X)$ be the corresponding representation of G on the vector space of complex-valued functions on X. Prove that $\mathscr{F}(X)$ contains the one-dimensional trivial representation of G with multiplicity $|G \setminus X|$, which is the number of G-orbits in X.

<u>Proof:</u> We know by part 1 of problem 7 from problem set 6 that $\mathscr{F}(X)$ has basis $(e_x)_{x\in X}$ where $e_x : X \to \mathbb{C}$ is given by $e_x(y) = \delta_{x,y}$ for all $y \in X$. Furthermore, If we let ρ be the representation of G on $\mathscr{F}(X)$, then we know from that same problem that for all $g \in G$ and $x \in X$, $\rho_g e_x = e_{gx}$, where the multiplication in the subscript of e_{gx} is given by the group action of G on X. Now let $n = |X|$, and let $x_1, \dots, x_n$ be any numbering of the elements of X. Since $\rho_g e_x = e_{gx}$ for all $g \in G$ and $x \in X$, we know that $\rho_g e_i = e_j$ iff $ge_i = e_j$. Therefore, given any $g \in G$, if we express ρ_g as a matrix A with entries a_{ij}, then the entries satisfy $a_{ij} = 1$ if $\rho_g e_j = e_i$, and $a_{ij} = 0$ otherwise. This observation allows us to compute

$$\begin{aligned}\chi_\rho(g) &= tr(\rho_g) = tr(A) = \sum_{i=1}^{n} a_{ii} = |\{1 \le i \le n : a_{ii} = 1\}| = |\{1 \le i \le n : \rho_g e_i = e_i\}| \\ &= |\{1 \le i \le n : gx_i = x_i\}| = |X^g|.\end{aligned}$$

Thus for all $g \in G$, $\chi_\rho(g) = |X^g|$.

Next, write $\mathscr{F}(X) = \bigoplus_{\lambda\in\Lambda} V_\lambda^{\oplus m_\lambda}$, and let ρ_λ be the irreducible representation corresponding to the subspace V_λ. Recall that for any irreducible representation ρ', $(\chi_{\rho_\lambda}, \chi_{\rho'}) = 1$ if $\rho_\lambda \simeq \rho'$ as representations, and $(\chi_{\rho_\lambda}, \chi_{\rho'}) = 0$ otherwise. Therefore, $(\chi_\rho, \chi_{\rho'}) = m_\lambda$ if $\rho_\lambda \simeq \rho'$, and $(\chi_\rho, \chi_{\rho'}) = 0$ if ρ' is not isomorphic to ρ_λ for any $\lambda \in \Lambda$. Also, by Burnside's lemma, $|G \setminus X| = |G|^{-1} \sum_{g\in G} |X^g|$. Next, note that the one-dimensional trivial representation $\rho_{\mathbb{1}}$ is such that for all $g \in G$, $\rho_{\mathbb{1}}(g)$ is given by the 1×1 matrix whose only entry is 1. Thus the character of the one-dimensional trivial representation satisfies $\chi_{\mathbb{1}}(g) = 1$ for all $g \in G$. For ease of notation, if $\rho_{\mathbb{1}} \simeq \rho_\lambda$ for some $\lambda \in \Lambda$, then define $m_{\mathbb{1}} = m_\lambda$, and otherwise define $m_{\mathbb{1}} = 0$. Thus $m_{\mathbb{1}}$ is precisely the number of irreducible subrepresentations of ρ which are isomorphic to $\rho_{\mathbb{1}}$. Combining all of these facts with our result from the previous paragraph allows us to compute

$$\begin{aligned}|G \setminus X| &= \frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\,\chi_{\mathbb{1}}\left(g^{-1}\right) = (\chi_\rho, \chi_{\mathbb{1}}) \\ &= \left(\sum_{\lambda\in\Lambda} m_\lambda \chi_{\rho_\lambda}, \chi_{\mathbb{1}}\right) = \sum_{\lambda\in\lambda} m_\lambda (\chi_{\rho_\lambda}, \chi_{\mathbb{1}}) = m_{\mathbb{1}}.\end{aligned}$$

Hence $m_{\mathbb{1}}$, which is the number of times that the representation ρ of G on $\mathscr{F}(X)$ contains the one-dimensional trivial representation $\mathbb{1}$ of G as a subrepresentation, is precisely equal to $|G \setminus X|$. This completes the proof.

Problem 8

Let λ be a partition of n, written as $\lambda = (1, \ldots, 1, 2, \ldots, 2, \ldots)$ where the index i is written l_i times for $i \in \mathbb{N}\,(\not\ni 0)$. Thus $\sum_{i\in\mathbb{N}} il_i = n$. Prove that the number of elements in the conjugacy class of S_n of cycle type λ is given by

$$\frac{n!}{\prod_{i\in\mathbb{N}} i^{l_i}\,(l_i!)}.$$

<u>Proof:</u> Suppose $\sigma \in S_n$ has cycle type λ. Since each index between 1 and n appears in precisely one cycle of σ, we know that there are $n!$ possible ways of writing σ. However, many of these ways of writing σ will yield the same permutation. To determine how many distinct choices exist for σ, fix some $i \in \mathbb{N}$. For any i-cycle $(a_1, \ldots, a_i)$, we know that

$$(a_1, \ldots, a_i) = (a_2, \ldots, a_i, a_1) = \cdots = (a_i, a_1, \ldots, a_{i-1})\,.$$

Therefore, the number of ways of writing any particular i-cycle is i. So for each i-cycle, we must divivde the number of choices for σ by i to account for the equivalent ways of writing each i-cycle. Since σ contains l_i i-cycles, we must divide the number of choices for σ by i^{l_i}. Furthermore, since the i-cycles are disjoint, reordering the i-cycles will have no impact on the value of σ. Since σ contains l_i i-cycles, there are $l_i!$ possible orderings of the i-cycles. To account for these reorderings, we must divide the number of choices for σ by i^{l_i}. Therefore, for each $i \in N$, we must divide the number of choices for σ by $i^{l_i}\,(l_i!)$. Once we do this for every $i \in \mathbb{N}$, we arrive at a final number of possible choices for σ given by

$$\frac{n!}{\prod_{i\in\mathbb{N}} i^{l_i}\,(l_i!)}.$$

Hence the above value is the number of elements in the conjugacy class of S_n consisting of elements of cycle type λ. This completes the proof.