ALGEBRA 1 HOMEWORK 8

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Problem 1. Compute the character table of S_5.

Solution: S_5 has the trivial representation V_{triv} (where the character is identically 1) and the sign representation V_{sgn} (where the character is the sign on the permutation). There is also a representation of S_5 on $\mathbb{C}^5$ given by permuting the basis elements, and we proved on the homework that this representation decomposes as the direct sum of the trivial representation and an irreducible representation which here will be called V_{std}. The character of V_{std} can be computed easily from the identity $\chi_{\mathbb{C}^5} = \chi_{\text{triv}} + \chi_{V_{\text{std}}}$ since $\chi_{\mathbb{C}^5}$ is just the number of fixed points of the input element of S_5. We get another irreducible representation for free by taking $V_{\text{std}} \otimes V_{\text{sgn}}$.

Let $V = \text{Ind}_{S_3 \times S_2}^{S_5}(\mathbb{1})$ where $\mathbb{1}$ is the trivial representation of $S_3 \times S_2$. The dimension of V is just the index of $S_3 \times S_2$ in S_5 which is $120/(6 \cdot 2) = 10$. As we did in class, we can use $\text{Ind}_H^G \chi_{\text{triv}}(g) = |(G/H)^g|$ to calculate the character of V First identify cosets of $S_5/(S_3 \times S_2)$ with two element subset of $\{1, \ldots, 5\}$, then count the number of such subsets that are fixed under the action of the coset representative. Then we can calculate $(\chi_V, \chi_V) = 3$, so there are three distinct (isomorphism types of) irreducible representations whose direct sum is V. Since $(\chi_{\text{triv}}, \chi_V) = (\chi_{V_1}, \chi_V) = 1$, both V_{triv} and V_1 appear in V with multiplicity 1. This implies that the remaining irreducible representation has dimension $10 - 4 - 1 = 5$, and we can calculate its character by subtracting the characters of V_{triv} and V_1 from the character of V. We get another irreducible representation for free here by taking $V_2 \otimes V_{\text{sgn}}$.

Since there are 7 conjugacy classes in S_5 and we have 6 of them, we can use that the sum of squares of the dimensions of irreducible representations is the size of the group to determine that the final representation has dimension 6. We call this final irreducible representation V_3. Since we know all the other characters, we can fill in the rest of the table by orthogonality. Putting it all together we get the following table:

	V_{triv}	V_{sgn}	V_{std}	$V_{\text{std}} \otimes V_{\text{sgn}}$	V_2	$V_2 \otimes V_{\text{sgn}}$	V_3
id	1	1	4	4	5	5	6
(12)	1	-1	2	-2	1	-1	0
(12)(34)	1	1	0	0	1	1	-2
(123)	1	1	1	1	-1	-1	0
(123)(45)	1	-1	-1	1	1	-1	0
(1234)	1	-1	0	0	-1	1	0
(12345)	1	1	-1	-1	0	0	1

Problem 2. Let W be the irreducible 2-dimensional representation of S_4 (see Lecture 19 page 4). Compute the decomposition of $\mathrm{Ind}_{S_4}^{S_5}(W)$ into a direct sum of irreducible representations of S_5. Here we view S_4 as a subgroup of S_5 consisting of permutations which fix the element 5.

Solution: First we compute the character of the induced representation using the formula

$$\mathrm{Ind}_{S_4}^{S_5}\chi_\pi(g) = \frac{1}{|S_4|}\sum_{x^{-1}gx\in S_4}\chi_\pi(x^{-1}gx)$$

where $\pi : S_4 \to GL(W)$ is the map corresponding to the representation W. Since W has dimension 2 and $(S_5 : S_4) = 5$, we know that $10 = \dim(\mathrm{Ind}_{S_4}^{S_5}W) = \mathrm{Ind}_{S_4}^{S_5}\chi_\pi(\mathrm{id})$. Conjugation doesn't change the cycle type or number of fixed points so if g has no fixed points, there are no valid x's to sum over so $\mathrm{Ind}_{S_4}^{S_5}\chi_\pi(g) = 0$ for $g = (12345)$ or $(123)(45)$. Furthermore if g fixes 5 and $\chi_\pi(g) = 0$, the sum will again be zero. In particular we get $\mathrm{Ind}_{S_4}^{S_5}\chi_\pi(g) = 0$ for $g = (12)$ and $g = (1234)$. $x \in S_5$ has $x^{-1}(12)(34)x \in S_4$ if and only if $x \in S_4$, in which case $\chi_\pi(x^{-1}(12)(34)x) = \chi_\pi((12)(34)) = 2$ since χ_π is a class function. Thus we get $\mathrm{Ind}_{S_4}^{S_5}\chi_\pi((12)(34)) = 2$. Finally to calculate the result on (123), note that $x^{-1}(123)x \in S_4$ if and only if $x(5) \in \{4, 5\}$. There are $2 \cdot |S_4| = 48$ such elements and just as before every term in the sum is equal to $\chi_\pi((123)) = -1$ so we obtain $\mathrm{Ind}_{S_4}^{S_5}\chi_\pi((123)) = -2$. Thus we have completely computed the character of this induced representation:

	id	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
$\mathrm{Ind}_{S_4}^{S_5}\chi_\pi$	10	0	2	-2	0	0	0

It is easy to check from the character table of S_5 that $\mathrm{Ind}_{S_4}^{S_5}\chi_\pi = \chi_{V_2} + \chi_{V_2\otimes V_{\mathrm{sgn}}}$, so it follows that

$$\mathrm{Ind}_{S_4}^{S_5}(W) = V_2 \oplus (V_2 \otimes V_{\mathrm{sgn}}).$$

Problem 4. Let V be the (unique) irreducible representation of S_5 of dimension 6 (from problem 1). Consider $S_3 \times S_2$ as a subgroup of S_5 consisting of permutations which preserve the decomposition $\{1, \dots, 5\} = \{1, 2, 3\} \cup \{4, 5\}$. Compute the decomposition of $\mathrm{Res}^{S_5}_{S_3 \times S_2}(V)$ into a direct sum of irreducible representations of $S_3 \times S_2$.

Solution: The following are the character tables for S_3 and S_2 respectively:

	V^3_{triv}	V^3_{sgn}	V^3_{std}
id	1	1	2
(12)	1	-1	0
(123)	1	1	-1

	V^2_{triv}	V^2_{sgn}
id	1	1
(45)	1	-1

where we have identified S_3 and S_2 as subsets of S_5 which act on $\{1, 2, 3\}$ and $\{4, 5\}$ respectively. By problem 5 from homework set 7, we can find all of the irreducible representations of $S_3 \times S_2$ by just taking tensor products of all pairs of irreducible representations of S_3 and S_2, and the conjugacy classes of $S_3 \times S_2$ are just pairs of conjugacy classes in S_3 and S_2:

	$V^3_{\text{triv}} \otimes V^2_{\text{triv}}$	$V^3_{\text{sgn}} \otimes V^2_{\text{triv}}$	$V^3_{\text{std}} \otimes V^2_{\text{triv}}$	$V^3_{\text{triv}} \otimes V^2_{\text{sgn}}$	$V^3_{\text{sgn}} \otimes V^2_{\text{sgn}}$	$V^3_{\text{std}} \otimes V^2_{\text{sgn}}$
id, id	1	1	2	1	1	2
(12), id	1	-1	0	1	-1	0
(123), id	1	1	-1	1	1	-1
id, (45)	1	1	2	-1	-1	-2
(12), (45)	1	-1	0	-1	1	0
(123), (45)	1	1	-1	-1	-1	1

The character of the restriction representation can be easily calculated by just considering the representative element of the conjugacy class in $S_3 \times S_2$ as an element of S_5 and using the character of the original representation on S_5:

	id, id	(12), id	(123), id	id, (45)	(12), (45)	(123), (45)
$\mathrm{Res}^{S_5}_{S_3 \times S_2} \chi_\pi$	6	0	0	0	-2	0

Now we just have to find the linear combination of the columns in the character table above that gives this character, which is the sum of the second, third, fourth, and sixth columns. Explicitly,

$$\mathrm{Res}^{S_5}_{S_3 \times S_2} \chi_\pi = (V^3_{\text{sgn}} \otimes V^2_{\text{triv}}) \oplus (V^3_{\text{std}} \otimes V^2_{\text{triv}}) \oplus (V^3_{\text{triv}} \otimes V^2_{\text{sgn}}) \oplus (V^3_{\text{std}} \otimes V^2_{\text{sgn}}).$$