Algebra Homework 9

Scott Newton

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Problem 3. With the set up of the previous problem, let W be the representation of S_4 defined by pullback of V under $S_4 \to S_4/(S_2 \times S_2) \tilde{\to} S_3$. Let $\tilde{V}$ be the standard irreducible 3-dimensional representation of S_4. Prove that

$$Ind_{S_2 \times S_2}^{S_4}(I) = I \oplus \tilde{V} \oplus W.$$

Proof. We can calculate $\chi_{Ind_{S_2 \times S_2}^{S_4} I}$ by $\chi_{Ind_{S_2 \times S_2}^{S_4} I}(g) = |(S_4/S_2 \times S_2)^g|$. So, counting fixed points for the cycle types in S_4 we see that the identity fixes every coset, of which there are $|S_4|/|S_2 \times S_2| = \frac{24}{4} = 6$.

Instead of considering the action on cosets, we can consider the action on 2 element subsets of $\{1, 2, 3, 4\}$ given by $\sigma.\{i, j\} = \{\sigma(i), \sigma(j)\}$ since there is an isomorphism between these actions.

A transposition (12) fixes both $\{1, 2\}$ and $\{3, 4\}$, so $\chi_{Ind_{S_2 \times S_2}^{S_4} I}((12)) = 2$.

For the element (123), no two element subsets are fixed, so $\chi_{Ind_{S_2 \times S_2}^{S_4} I}((123)) = 0$.

For the lement (12)(34), both $\{1, 2\}$ and $\{3, 4\}$ are fixed, so $\chi_{Ind_{S_2 \times S_2}^{S_4} I}((12)(34)) = 2$.

For an element (1234), no two element subsets are fixed, so $\chi_{Ind_{S_2 \times S_2}^{S_4} I}((1234)) = 0$.

Thus, we get the following character:

conj. class	e	(12)	(12)(34)	(123)	(1234)
# elts	1	6	3	8	6
$\chi_{Ind_{S_2 \times S_2}^{S_4} I}$	6	2	2	0	0

Recall,

conj. class	e	(12)	(12)(34)	(123)	(1234)
# elts	1	6	3	8	6
$\chi_{\tilde{V}}$	3	1	-1	0	-1
χ_{triv}	1	1	1	1	1

Taking the following inner products, we confirm the decomposition into irreducibles:

$(\chi_{Ind_{S_2 \times S_2}^{S_4} I}, \chi_{triv}) = \frac{1}{24}(6 + 6 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1) = 1$.

$(\chi_{Ind_{S_2 \times S_2}^{S_4} I}, \chi_{\tilde{V}}) = \frac{1}{24}(6 \cdot 3 + 6 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot (-1)) = 1$.

This leaves the final character,

conj. class	e	(12)	(12)(34)	(123)	(1234)
# elts	1	6	3	8	6
$\chi_{Ind_{S_2\times S_2}^{S_4} I} - \chi_{\tilde{V}} - \chi_{triv}$	2	0	2	-1	0

We see that this is irreducible because $(\chi_{Ind_{S_2\times S_2}^{S_4} I} - \chi_{\tilde{V}} - \chi_{triv}, \chi_{Ind_{S_2\times S_2}^{S_4} I} - \chi_{\tilde{V}} - \chi_{triv}) = 1$.

We identify $S_2 \times S_2$ with the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\} < S_4$ in order to define W as the pullback of the standard 2-dim irreducible representation of S_3 via $S_4 \to S_4/(S_2 \times S_2) \tilde{\to} S_3$.

Recall,

conj. class	e	(12)	(123)
χ_{std}	2	0	-1

So, pulling back to S_4 and considering its conjugacy classes, we find $\chi_W(e) = 2$, $\chi_W((12)) = \chi_W((12)(34)) = 0$, and $\chi_W((123)) = -1$. Hence, we have the decomposition $Ind_{S_2\times S_2}^{S_4}(I) = I \oplus \tilde{V} \oplus W$.

$\square$

Problem 4. Let G be a finite group and $f \in (CG)_{class}$. Prove that f is a character of an irreducible finite-dimensional representation of G if, and only if the following three conditions are satised:

(i) f is a $\mathbb{Z}$-linear combination of the characters of some finite-dimensional representations of G.

(ii) $(f, f) = 1$.

(iii) $f(e) > 0$ where e is the identity element of G.

Proof. Let f be a character of an irreducible finite-dimensional representation of G. Then f is trivially a $\mathbb{Z}$-linear combination of characters of finite-dimensional representations of G. Also, we know that $(f, f) = 1$ because f is the character of an irreducible finite-dimensional represenation. Lastly, irreducible finite dimensional representations have positive dimension and $f(e)$ is the dimension of the irreducible whose character is f, so $f(e) > 0$.

Now assume that f satisfies properties (i), (ii), and (iii). Then, $f = \sum_{i=1}^{k} a_i \chi_i$ is a $\mathbb{Z}$-linear combination of characters. If these character are not irreducible, then we can decompose them further until we have a linear combination of characters of irreducibles. Hence, without loss of generality, we may assum that the χ_i are characters of irreducible representations. Thus,

$$
\begin{aligned}
1 = (f,f) &= (\sum_{i=1}^{k} a_i\chi_i, \sum_{j=1}^{k} a_j\chi_j) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j (\chi_i,\chi_j) \\
&= \sum_{i=1}^{k} a_i^2,
\end{aligned}
$$

Hence, exactly one a_i has magnitude exactly 1. Let's call this a_1 without loss of generality. Since $f(e) > 0$, and $\chi_1(e) > 0$, we must have $a_1 = 1$. Therefore, $f = \sum_{i=1}^{k} a_i\chi_i = \chi_1$, so f is the character of an irreducible finite-dimensional representation of G. □

Problem 5. For each partition λ of 5, let $\iota_\lambda \in (CS_5)_{class}$ be the character of the induced representation $Ind_{S_\lambda}^{S_5}(I)$. Prove that the following class function is the character of some finite-dimensional irreducible representation of S_5:

$$s_{(3,2)} := \iota_{(3,2)} - \iota_{(4,1)}$$

Proof. We previously computed the following character table for S_5:

conj. class	#	χ_{triv}	χ_{sign}	χ_{std}	$\chi_{std\otimes sign}$	$\chi_{V-std-triv}$	$\chi_{(V-std-triv)\otimes sign}$	χ_{last}
(1^5)	1	1	1	4	4	5	5	6
$(1^3,2^1)$	10	1	-1	2	-2	1	-1	0
$(1^1,2^2)$	15	1	1	0	0	1	1	-2
$(1^2,3^1)$	20	1	1	1	1	-1	-1	0
$(2^1,3^1)$	20	1	-1	-1	1	1	-1	0
$(1^1,4^1)$	30	1	-1	0	0	-1	1	0
(5^1)	24	1	1	-1	-1	0	0	1

Restricting to $S_3 \times S_2 < S_5$, we get the following characters:

conj. class	#	χ_{triv}	χ_{sign}	χ_{std}	$\chi_{std\otimes sign}$	$\chi_{V-std-triv}$	$\chi_{(V-std-triv)\otimes sign}$	χ_{last}
e	1	1	1	4	4	5	5	6
$(12)\times e$	3	1	-1	2	-2	1	-1	0
$e\times(45)$	1	1	-1	2	-2	1	-1	0
$(12)\times(45)$	3	1	1	0	0	1	1	-2
$(123)\times e$	2	1	1	1	1	-1	-1	0
$(123)\times(45)$	2	1	-1	-1	1	1	-1	0

Thus, we can compute the character of $\iota_{(3,2)} = Ind_{S_3\times S_2}(I)$ via Frobenius Reciprocity: $(\iota_{(3,2)},\chi_i) = (\chi_{triv}, Res_{S_3\times S_2}^{S_5}(\chi_i))$ for each character χ_i of an irreducible .
$(\chi_{triv}, Res_{S_3\times S_2}^{S_5}(\chi_{triv})) = \frac{1}{12}(1+3\cdot 1+1+3\cdot 1+2\cdot 1+2\cdot 1) = 1.$

$(\chi_{triv}, Res^{S_5}_{S_3\times S_2}(\chi_{sign})) = \frac{1}{12}(1 - 3\cdot 1 - 1 + 3\cdot 1 + 2\cdot 1 - 2\cdot 1) = 0.$
$(\chi_{triv}, Res^{S_5}_{S_3\times S_2}(\chi_{std})) = \frac{1}{12}(4 + 3\cdot 2 + 2 + 3\cdot 0 + 2\cdot 1 - 2\cdot 1) = 1.$
$(\chi_{triv}, Res^{S_5}_{S_3\times S_2}(\chi_{std\otimes sign})) = \frac{1}{12}(4 - 3\cdot 2 - 2 + 3\cdot 0 + 2\cdot 1 + 2\cdot 1) = 0.$
$(\chi_{triv}, Res^{S_5}_{S_3\times S_2}(\chi_{V-std-triv})) = \frac{1}{12}(5 + 3\cdot 1 + 1 + 3\cdot 1 - 2\cdot 1 + 2\cdot 1) = 1.$
$(\chi_{triv}, Res^{S_5}_{S_3\times S_2}(\chi_{(V-std-triv)\otimes sign})) = \frac{1}{12}(5 - 3\cdot 1 - 1 + 3\cdot 1 - 2\cdot 1 - 2\cdot 1) = 0.$

So, $\iota_{(3,2)} = (\chi_{triv}) \oplus \chi_{std} \oplus \chi_{V-std-triv}$, so:

conj. class	#	$\iota_{(3,2)}$
(1^5)	1	10
$(1^3, 2^1)$	10	4
$(1^1, 2^2)$	15	2
$(1^2, 3^1)$	20	1
$(2^1, 3^1)$	20	1
$(1^1, 4^1)$	30	0
(5^1)	24	0

Next, we calculate $\iota_{(4,1)}$ as the number of fixed points $\iota_{(4,1)}(g) = |(S_5/S_4)^g|$:

conj. class	#	$\iota_{(4,1)}$
(1^5)	1	5
$(1^3, 2^1)$	10	3
$(1^1, 2^2)$	15	1
$(1^2, 3^1)$	20	2
$(2^1, 3^1)$	20	0
$(1^1, 4^1)$	30	1
(5^1)	24	0

Thus, $s_{(3,2)} = \iota_{(3,2)} - \iota_{(4,1)}$:

conj. class	#	$s_{(3,2)}$
(1^5)	1	5
$(1^3, 2^1)$	10	1
$(1^1, 2^2)$	15	1
$(1^2, 3^1)$	20	-1
$(2^1, 3^1)$	20	1
$(1^1, 4^1)$	30	-1
(5^1)	24	0

This is indeed a character of one of the irreducibles of S_5 from the table above.

$\square$