ALGEBRA 1. PROBLEM SET 1

Problem 1. Let G and G' be two groups and let $\varphi : G \to G'$ be a group homomorphism. Prove that

(1) φ is injective if, and only if $\mathrm{Ker}(\varphi) = \{e\}$.

(2) φ is surjective if, and only if $\mathrm{Im}(\varphi) = G'$.

(3) φ is an isomorphism if, and only if it is bijective (as a set map).

Problem 2. Let G be a finite group of order p a prime number. Prove that every non–identity element of G is a generator of G. In particular, G is cyclic and hence abelian.

Problem 3. Let G be a group and H_1, H_2 two subgroups of G. Assume $G = H_1 \cup H_2$. Prove that $G = H_1$ or $G = H_2$.

Problem 4. Let G be a group and H be a non–empty subset of G. Prove that H is a subgroup of G if, and only if for every $x, y \in H$, $x^{-1}y \in H$.

Problem 5. Let G be a group and H_1, H_2 two subgroups of G of finite index. Prove that $(G : H_1 \cap H_2) < \infty$.

Problem 6. Is $\mathbb{Q}$ (rational numbers : abelian group under addition) finitely generated? Prove that there is no subgroup H of $\mathbb{Q}$ for which $(\mathbb{Q} : H) < \infty$.

Problem 7. Let G be a group of even order and let H be a subgroup of G such that $(G : H) = 2$. Prove that H is normal.

Problem 8. Let G be a group such that every non–identity element is of order 2. Prove that G is abelian.

Problem 9. Let m, n be two positive integers. What is the cardinality of

$\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) :=$ the set of all group homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$?

Problem 10. Let n be a positive integer. What is the order of the group $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$: the group of all isomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to itself?

Problem 11. Let G be a group and consider the following subset of G:

$$X := \{aba^{-1}b^{-1} : a, b \in G\}$$

Let $H = \langle X \rangle$ be the subgroup generated by X. Prove that:

(1) H is a normal subgroup.

(2) G/H is abelian.

Problem 12. Let G be a group and $g \in G$. Define $C_g : G \to G$ by $C_g(x) = gxg^{-1}$. Prove that

(1) C_g is an isomorphism.

(2) $C : G \to \mathrm{Aut}(G)$ defined by $g \mapsto C_g$ is a group homomorphism.

(3) The image of C in $\mathrm{Aut}(G)$ is a normal subgroup (called the group of inner automorphisms usually denoted by $\mathrm{Inn}(G)$).

Problem 13. Let G be a finite abelian group, written additively below. Let $x = \sum_{g \in G} g$. Let $H \subset G$ be defined by: $H = \{g \in G : 2g = 0\}$. Prove that

(1) $x = \sum_{h \in H} h$.

(2) If $|H| \neq 2$ then $x = 0$. If $|H| = 2$ then $H = \{e, x\}$.

Problem 14. Let p be a prime number and let $G = (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ be the group under multiplication. Use the previous exercise to prove that $(p-1)! \equiv -1 \pmod{p}$.

Problem 15. Let G be a group acting on a set X. Assume that the action is free and transitive. Pick $x \in X$ and define a set map $G \to X$ by $g \mapsto gx$. Prove that this map is bijective.

Problem 16. Consider the following group acting on $X = \mathbb{R}^2 \setminus \{(0,0)\}$

$$G = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \leq \theta \leq 2\pi \right\}$$

Determine whether the action of G on X is free, faithful, transitive. Describe the orbit space $G \backslash X$.

Problem 17. Assume a finite group G acts on a finite set X transitively. Assume that $|X| \geq 2$. Prove that there exists $g \in G$ such that $X^g = \emptyset$.

Problem 18. Let G be a group and H be a subgroup of G. Consider the action of G on G/H by: $x \cdot (gH) = xgH$.

(1) What is the stabilizer of a left coset $gH \in G/H$?

(2) Prove that this action is faithful if, and only if $\cap_{g \in G} gHg^{-1} = \{e\}$.

Problem 19. Assume G is a group and H is a subgroup such that $(G : H) < \infty$. Prove that there exists a normal subgroup N of G such that $(G : N) < \infty$ and $N \subset H$. (Hint: consider G acting on the finite set G/H).

Problem 20. Let G be the group of symmetries of a regular hexagon. What is the order of G?