ALGEBRA 1. PROBLEM SET 2

Problem 1. Let $\varphi : G_1 \to G_2$ be a group homomorphism. Assume that G_2 is abelian. Prove that φ factors through $N = \langle X \rangle$: the (normal) subgroup generated by the following set of elements (see Problem 11 of Problem Set 1):

$$X := \{aba^{-1}b^{-1} : a, b \in G_1\}$$

Problem 2. Let G be a finite group and H, N be two normal subgroups of G. Assume that $|H|$ and $|N|$ are coprime. Prove that $hx = xh$ for every $h \in H$ and $x \in N$. Prove that $H \cap N = \{e\}$. Prove that the subgroup of G generated by H and N is isomorphic to the direct product $H \times N$.

Problem 3. Let $C : G \to \mathrm{Aut}(G)$ be given by $g \mapsto C_g$ where $C_g(x) = gxg^{-1}$ (see Problem 12 of Problem Set 1). Is $G \ltimes_C G$ isomorphic to $G \times G$?

Problem 4. Let A, B be two groups and $G = A \times B$. Let H be a subgroup of G such that $A \subset H$. Prove that $H = A \times (H \cap B)$.

Problem 5. Let G be a group and let $Z(G)$ be its center. That is, $Z(G) = \{x \in G : gx = xg \text{ for every } g \in G\}$.

(1) Prove that $Z(G)$ is a normal subgroup of G.

(2) Assume that there is a subgroup $H \subset Z(G)$ such that G/H is cyclic. Prove that G is abelian.

Problem 6. Assume that there is a short exact sequence of group homomorphisms:

$$\mathbf{1} \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} \mathbb{Z} \longrightarrow \mathbf{1}$$

Further assume that $\mathrm{Im}(\varphi) \subset Z(B)$. Prove that this exact sequence is trivial (in particular, $B = A \times \mathbb{Z}$).

Problem 7. Let A_1 and A_2 be two groups and G be a subgroup of $A_1 \times A_2$. Let $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ be the natural projections. Define:

$$N_1 = G \cap A_1; \qquad H_1 = \pi_1(G)$$
$$N_2 = G \cap A_2; \qquad H_2 = \pi_2(G)$$

Prove that N_1 is normal in H_1 and N_2 is normal in H_2. Prove that there exists an isomorphism $H_1/N_1 \to H_2/N_2$.

Problem 8. For each of the following short exact sequences, determine whether it is split and/or trivial. In each case write a section and/or a retraction. Are sections/retractions unique?

(1) Recall that $SL_2(\mathbb{C})$ is the group of 2×2 matrices of determinant 1, and $GL_2(\mathbb{C})$ is the group of invertible 2×2 matrices (with entries from the field

of complex numbers). The following is the short exact sequence associated to determinant $\det : GL_2(C) \to \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

$$\mathbf{1} \longrightarrow SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^\times \longrightarrow \mathbf{1}$$

(2) Consider the natural inclusion of $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$ and the following short exact sequence arising from it.

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

(3) Recall that S_n is the group of permutations of $\{1, \ldots, n\}$. Let $\varepsilon : S_3 \to \{\pm 1\}$ be the sign homomorphism.

$$\mathbf{1} \longrightarrow A_3 \longrightarrow S_3 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{1}$$

Problem 9. How many (up to isomorphism) groups are there which will fit in the following short exact sequence?

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Problem 10. Consider the following set of elements in $A_4 \subset S_4$.

$$\{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$$

(1) Prove that they form a normal subgroup in S_4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(2) Prove that the following short exact sequence splits.

$$\mathbf{1} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow A_4 \longrightarrow A_3 \longrightarrow \mathbf{1}$$

(3) Does the following short exact sequence split?

$$\mathbf{1} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow \mathbf{1}$$