ALGEBRA 1. PROBLEM SET 3

For this problem set, p will denote a prime number.
All groups considered here are finite, except Problem 1.

Problem 1. Let A, B, C be three abelian groups and let there be a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Prove that if this sequence splits, then it is trivial.

Problem 2. Let G be a group and $H < G$ be a subgroup. Let $P < H$ be a Sylow p–subgroup of H. Prove that there exists a Sylow p–subgroup of G, say Q, such that $P = Q \cap H$.

Problem 3. Let G be a group and $N \triangleleft G$ be a normal subgroup. Consider $\pi : G \to G/N$ the natural surjection.

(1) Let P be a Sylow p–subgroup of G. Prove that $P \cap N$ is a Sylow p–subgroup of N.

(2) Prove that $\pi(P)$ is a Sylow p–subgroup of G/N.

Problem 4. Let $P < G$ be a Sylow p–subgroup of G, and let $N = N_G(P)$ be the normalizer of P (that is, $N = \{g \in G : gPg^{-1} = P\}$). Prove that for every $L < G$ containing N, $N_G(L) = L$.

Problem 5. Let $H \triangleleft G$ be a normal subgroup of a group G. Let us assume that $|H| = p$. Prove that H is contained in every Sylow p–subgroup of G.

Problem 6. Let $K \triangleleft G$ be a normal subgroup of a group G, and let P be a Sylow p–subgroup of K. Prove that $G = K \cdot N_G(P)$.

Problem 7. Let $G = GL_N(\mathbb{F}_p)$ where $N \in \mathbb{Z}_{\geq 2}$. What is the order of G? Describe a Sylow p–subgroup of G.

Problem 8. Assume G is a group of order $2p$ (assume $p \neq 2$ for this problem). Prove that either G is cyclic or the dihedral group D_p.

Problem 9. Assume that G is a non–abelian group of order p^3.

(1) Prove that $Z(G) \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$.

(2) Prove that $G/Z(G) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^2$.

(3) Prove that every subgroup of G, of order p^2, is normal and contains $Z(G)$.

Problem 10. Let q be another prime number and $p < q$. Assume that G is a group of order pq.

(1) Prove that G is not simple.

(2) Further assume that $q \not\equiv 1 \pmod{p}$. Prove that G is cyclic.

Problem 11. Prove that there is no simple group of order p^2q, where p, q are prime numbers (not necessarily distinct).

Problem 12. Let $a \in \{1, \ldots, p-1\}$ and $e \geq 1$. Prove that there is no simple group of order $p^e a$.

Problem 13. Let G be a non–abelian group of order 6. Prove that G is isomorphic to S_3.

Problem 14. How many (up to isomorphism) groups of order 18 are there?

Problem 15. For each of the following numbers, prove that there is no simple group of that order: (a) 12; (b) 40; (c) 216; (d) 224.

Problem 16. Describe a Sylow 2–subgroup of the dihedral group D_{10} of order 20.

Problem 17. Let G be a simple group of order 60 (assume it exists!). How many Sylow q–subgroups are there in G, for $q = 2, 3$ and 5?