ALGEBRA 1. PROBLEM SET 4

Problem 1. (Fun with commutators) Let G be a group. For $a, b \in G$, define $(a, b) := aba^{-1}b^{-1}$. Recall that for any two subsets $A, B \subset G$, we defined (A, B) to be the subgroup generated by $\{(a, b) : a \in A, b \in B\}$.

(1) Verify the following identity, for all $a, x, y \in G$.
$$(a, xy) = (a, x)(x, (a, y))(y, a)$$
(2) Let A, B, C be three normal subgroups of G. Prove that $(A, (B, C))$ is generated by $\{(a, (b, c)) : a \in A, b \in B, c \in C\}$.

(3) Recall that $C^1(G) = G$ and $C^{n+1}(G) = (G, C^n(G))$. Prove that for every $m, n \geq 1$ we have
$$(C^m(G), C^n(G)) \subset C^{m+n}(G)$$
(4) Recall that $D^0(G) = G$ and $D^{n+1}(G) = (D^n(G), D^n(G))$. Prove that $D^l(G) \subset C^{2^l}(G)$, for every $l \geq 0$.

Problem 2. Prove that every nilpotent group is solvable. **Do not** use the results of the previous problem.

Problem 3. Consider the group of 2×2 upper triangular matrices:
$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{C}^\times; \quad b \in \mathbb{C} \right\}$$
Prove that B is solvable, but not nilpotent.

Problem 4. Let G be a group and let N_1, N_2 be two normal subgroups. Prove that if $N_1 \cap N_2 = \{e\}$ then $xy = yx$ for every $x \in N_1$ and $y \in N_2$.

Problem 5. Let N_1, N_2 be two normal subgroups of G, such that $(G, N_1) \subset N_2 \subset N_1$. For any subgroup H of G, prove that N_2H is normal in N_1H.

Problem 6. Prove that the following three assertions about a finite group G are equivalent.

(1) G is nilpotent.

(2) Every Sylow subgroup of G is normal.

(3) G is a direct product of p–groups.

[Hints: Lemma 10.10 (p. 9 of Lecture 10); and Problem 4 of Set 3]

Problem 7. Let $\varphi : G \to G'$ be a group homomorphism. Assume Σ' is a composition series of G':
$$\Sigma' : G' = G'_0 \triangleright G'_1 \triangleright \ldots \triangleright G'_n = \{e\}$$

Let Σ be the sequence with terms $G_j = \varphi^{-1}(G'_j)$.

(1) Prove that Σ is a composition series of G.

(2) Prove that we have injective homomorphisms $\mathrm{gr}_i^{\Sigma}(G) \to \mathrm{gr}_i^{\Sigma'}(G')$ for each $0 \leq i \leq n-1$.

Problem 8. For a group H that has a Jordan–Hölder series, let $\ell(H)$ denote the number of terms in a JH series of H. Now let G be a group and N be a normal subgroup of G. Prove that G has a Jordan–Hölder series if, and only if both N and G/N do. In that case, prove that $\ell(G) = \ell(N) + \ell(G/N)$.

Problem 9. Let G be a nilpotent group and let H be a proper subgroup of G. Prove that there exists a proper normal subgroup N of G, which contains H and such that G/N is abelian.

Problem 10. Again let G be a nilpotent group and H be a subgroup. Prove that if $G = H.(G,G)$ then $H = G$. In other words, a subset X of G generates G if, and only if the image of X under the natural surjection generates $G/(G,G)$.

[Hint for Problems 9, 10: see the proof of Lemma 10.10 (p. 9 of Lecture 10)]