ALGEBRA 1. PROBLEM SET 5

Notations: S_n is the group of permutations on n letters $\{1, \dots, n\}$. For $1 \leq i \leq n-1$, $s_i = (i\ i+1)$ denotes the simple transposition.

Problem 1. Recall that we defined $\ell(\pi)$ for a permutation $\pi \in S_n$ as the smallest number ℓ such that π can be written as a product of ℓ simple transpositions. Prove that $\ell(\pi s_k) < \ell(\pi)$ if, and only if $\pi(k) > \pi(k+1)$.

Problem 2. Prove that $\ell(\pi)$ is same as the cardinality of the following set

$$\{(i,j) : 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\}$$

Problem 3. Let G_n be the group given by the following presentation: G_n has $n-1$ generators $g_1, \dots, g_{n-1}$ and these generators satisfy the following list of relations:

$$g_i^2 = e \text{ for every } 1 \leq i \leq n-1$$
$$g_i g_j = g_j g_i \text{ for every } i, j \text{ such that } |i-j| > 1$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for every } 1 \leq i \leq n-2$$

- Prove that there is a unique surjective group homomorphism $G_n \to S_n$ sending g_i to s_i.
- Let H be the subgroup of G_n generated by $g_1, \dots, g_{n-2}$. Prove that the following is the list of all cosets G_n/H:
$$H_0 = H;\ H_1 = g_{n-1}H;\ H_2 = g_{n-2}H_1 = g_{n-2}g_{n-1}H; \cdots$$
$$H_{n-1} = g_1 H_{n-1} = g_1 \cdots g_{n-1} H$$
- Prove by induction on n that $|G_n| \leq n!$. Hence $G_n \xrightarrow{\sim} S_n$.

Problem 4. Determine the conjugacy classes in S_5 and the number of elements in each class. Then determine all Sylow subgroups of S_5.