ALGEBRA 1. PROBLEM SET 6

Notation. All vector spaces considered below are over $\mathbb{C}$ the field of ocmplex numbers.

Problem 1. Let V be an m–dimensional vector space and assume $f \in \mathrm{End}_{\mathbb{C}}(V)$. Let us choose a basis of V, say $\{e_1, \ldots, e_m\}$, and express f as an $m \times m$–matrix, $A = (a_{ij})$ whose entries are determined as follows:

$$f(e_i) = \sum_{j=1}^{m} a_{ji} e_j \text{ for every } 1 \leq i, j \leq m$$

Consider the natural linear map $\wedge^\ell(f) : \bigwedge^\ell(V) \to \bigwedge^\ell(V)$. Verify that the coefficient of the basis vector $e_{i_1} \wedge \cdots \wedge e_{i_\ell}$ in $\wedge^\ell(f)(e_{j_1} \wedge \cdots \wedge e_{j_\ell})$ is given by

$$\sum_{\sigma \in S_\ell} \mathrm{sign}(\sigma) a_{i_{\sigma(1)}, j_1} \cdots a_{i_{\sigma(\ell)}, j_\ell}$$

Here $1 \leq i_1 < \ldots i_\ell \leq m$ and $1 \leq j_1 < \ldots < j_\ell \leq m$.

Problem 2. Let V be an m–dimensional vector space. Prove that

$$\dim\left(\mathrm{Sym}^\ell(V)\right) = \begin{pmatrix} m + \ell - 1 \\ \ell \end{pmatrix}$$

Problem 3. Let G be a group and V be a representation of G. Prove that we have the following decomposition as representations of G:

$$\mathrm{Sym}^2(V) \oplus \bigwedge^2(V) \xrightarrow{\sim} V \otimes V$$

Problem 4. Let G be a group and V, W be two representations of G. Recall that we have a natural map $\varphi : V^* \otimes W \to \mathrm{Hom}_{\mathbb{C}}(V, W)$. Prove that φ is a G–intertwiner.

Problem 5. Let V, W be two representations of a group G. Prove that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^G$. (Recall for a set X with a G–action, $X^G := \{x \in X : gx = x \, \forall g \in G\}$).

Problem 6. Let V be a representation of a group G. Assume there exists $P \in \mathrm{Hom}_G(V, V)$ such that $P^2 = P$. Prove that $\mathrm{Ker}(P) \oplus \mathrm{Im}(P) \xrightarrow{\sim} V$ (as representations of G).

Problem 7. Let G be a group acting on a finite set X. Recall that we have a representation of G on the vector space of $\mathbb{C}$–valued functions on X, denoted by $\mathrm{Fun}(X)$.

(1) Consider the following basis of X. For every $x \in X$, let $\varepsilon_x : X \to \mathbb{C}$ be the function $\varepsilon_x(y) = \delta_{x,y}$ ($= 1$ if $x = y$; 0 otherwise). Prove that $g \cdot \varepsilon_x = \varepsilon_{gx}$ for every $g \in G$ and $x \in X$.

(2) Prove that for every orbit $\alpha \in G\backslash X$ we have a subrepresentation of $\mathrm{Fun}(X)$ given by the span of $\{\varepsilon_x : x \in \alpha\}$.

(3) Assume $|X| \geq 2$. Prove that $\mathrm{Fun}(X)$ is never irreducible.

Problem 8. Let V be a vector space. A Hermitian inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that

- $\langle z_1 v_1 + z_2 v_2, v \rangle = z_1 \langle v_1, v \rangle + z_2 \langle v_2, v \rangle$ for every $v_1, v_2, v \in V$ and $z_1, z_2 \in \mathbb{C}$.
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for every $v, w \in V$.
- $\langle v, v \rangle > 0$ for every $v \in V$.

Assume that V is a finite–dimensional representation of a finite group G. Prove that V has a G–invariant Hermitian inner product. The word G–invariant means $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$ for every $g \in G$ and $v, w \in V$.

Problem 9. Use the previous problem to reprove the following result. Let V be a finite–dimensional representation of a finite group G. Let $V_1 \subset V$ be a subrepresentation. Then V_1 has a G–stable complementary subspace, i.e, there exists a subrepresentation $V_2 \subset V$ such that $V \xrightarrow{\sim} V_1 \oplus V_2$ as representations of G.

Problem 10. Let G be a group. Prove that the set of 1–dimensional representations of G is in bijection with the set of group homomorphisms from $G/(G,G)$ to $\mathbb{C}^\times$ (the set of non–zero complex numbers, considered to be a group under multiplication).

Problem 11. Assume $G = \mathbb{Z}/n\mathbb{Z}$. Prove that the set of 1–dimensional representations of G is in bijection with the set of n^{th} roots of unity, i.e, $\{e^{2\pi\iota k/n} : 0 \leq k \leq n-1\}$.