ALGEBRA 1. PROBLEM SET 7

Unless otherwise stated, the groups considered below are finite, and vector spaces are over $\mathbb{C}$.

Problem 1. Recall that we have the following isomorphisms of vector spaces:

$$\mathbb{C}G \xrightarrow{\quad\Psi\quad} \bigoplus_{\lambda\in\Lambda_G} \mathrm{End}_{\mathbb{C}}(V_\lambda) \xleftarrow{\quad\Phi\quad} \bigoplus_{\lambda\in\Lambda_G} V_\lambda^* \otimes V_\lambda$$

where Λ_G is the set of isomorphism classes of finite–dimensional irreducible representations of G.

(1) Prove that the map $V_\lambda^* \otimes V_\lambda \to \mathbb{C}G$ given by restricting the composition $\Psi^{-1} \circ \Phi$ to the λ^{th} summand, is given by the matrix coefficients:
$$\xi \otimes v \mapsto \{g \mapsto \xi(g \cdot v)\}$$
(2) Consider the $G \times G$ action on $\mathbb{C}G$ given by the following formula, for every $g_1, g_2, x \in G$ and $f \in \mathbb{C}G$:
$$[(g_1, g_2) \cdot f]\,(x) = f(g_1^{-1} x g_2)$$
Prove that $\Phi^{-1} \circ \Psi$ is a $G \times G$–intertwiner; where $G \times G$ acts on $V_\lambda^* \otimes V_\lambda$ by $(g_1, g_2) \cdot (\xi \otimes v) = (g_1 \cdot \xi) \otimes (g_2 \cdot v)$.

(3) Define a map $* : \mathbb{C}G \times \mathbb{C}G \to \mathbb{C}G$ as follows. For every $f_1, f_2 \in \mathbb{C}G$:
$$f_1 * f_2(x) = \sum_{g\in G} f_1(xg^{-1}) f_2(g)$$
Prove that $\Psi(f_1 * f_2) = \Psi(f_1) \circ \Psi(f_2)$ where the operation on the right–hand side is composition of linear endomorphisms.

Problem 2. Prove that a finite group is abelian if, and only if all its irreducible finite–dimensional representations are one–dimensional.

Problem 3. Consider the action of the symmetric group S_n on $V \subset \mathbb{C}^n$ by permutation of coordinates, where,
$$V := \left\{ \sum_{i=1}^n a_i \varepsilon_i : a_1 + \cdots + a_n = 0 \right\}$$
Prove that V is an irreducible S_n–representation.

Problem 4. Recall that D_n is the group generated by two elements s_1, s_2 subject to the relations: $s_1^2 = s_2^2 = (s_1 s_2)^n = 1$. Consider the action of the dihedral group D_n on $\mathbb{C}^2$ given by the group homomorphism $D_n \to GL_2(\mathbb{C})$:
$$s_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad\qquad s_2 \mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$
where $\theta = \dfrac{2\pi}{n}$. Prove that for $n \geq 3$ the resulting representation is irreducible.

Problem 5. Let G and H be two groups and let $\{V_\lambda\}_{\lambda\in\Lambda_G}$ and $\{W_\mu\}_{\mu\in\Lambda_H}$ be the sets of all finite–dimensional irreducible representations of G and H respectively. Prove that $\{V_\lambda \otimes W_\mu\}_{\lambda\in\Lambda_G,\mu\in\Lambda_H}$ is the complete list of finite–dimensional irreducible representations of $G\times H$.

Problem 6. Let G be a group acting on a finite (non–empty) set X. Let $\mathrm{Fun}(X)$ be the corresponding representation of G on the vector space of complex–valued functions on X. Prove that $\mathrm{Fun}(X)$ contains the one–dimensional trivial representation of G with multiplicity $|G\backslash X|$ (the number of G–orbits in X).

Problem 7. Let V be a finite–dimensional representation of G such that $\chi_V(g) = 0$ for every $g \neq 1$ (g is not the identity element of G). Prove that $|G|$ divides $\dim(V)$.

Problem 8. Let λ be a partition of n, written as

$$\lambda = (\underbrace{1,\ldots,1}_{\ell_1 \text{ times}},\underbrace{2,\ldots,2}_{\ell_2 \text{ times}},\ldots)$$

(thus $\sum_{i\geq 1} i.\ell_i = n$). Prove that the number of elements in the conjugacy class of S_n of cycle–type λ is given by $\dfrac{n!}{\prod_{i\geq 1} i^{\ell_i}(\ell_i!)}$.

Problem 9. Let $H < G$ be an *abelian* subgroup. Let V be an irreducible finite–dimensional representation of G. Prove that $\dim(V) \leq |G/H|$.

Problem 10. Compute the character table of the dihedral group D_4.

Problem 11. Let Q_8 be the group with 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by

$$i^2 = j^2 = k^2 = 1$$
$$i \cdot j = k;\; j \cdot i = -k;\; j \cdot k = i;\; k \cdot j = -i;\; k \cdot i = j;\; i \cdot k = -j$$

Compute the character table of Q_8 and verify that both D_4 and Q_8 have the same character table (up to reordering conjugacy classes).