## ALGEBRA 1: PROBLEM SET 9

**Problem 1.** Let $G$ be a finite group and $H < G$ be a subgroup. Let $\{g_1, \ldots, g_\ell\}$ be the representatives of cosets in $G/H$ (that is, $G = \sqcup_{i=1}^{\ell} g_i H$). The left action of $G$ on the set $G/H$ gives us a map $\sigma : G \to S_\ell$ as follows. For every $x \in G$ and $1 \leq i \leq \ell$, $\sigma(x)(i)$ is the unique element of $\{1, \ldots, \ell\}$ such that $g_{\sigma(x)(i)}^{-1} x g_i \in H$.

Let $\pi : H \to \mathrm{GL}(W)$ be a finite–dimensional representation of $H$. Define a representation of $G$ on the vector space $W \otimes \mathbb{C}^\ell$ by the following formula, where $x \in G$, $w_1, \ldots, w_\ell \in W$ and $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$ is the usual basis of $\mathbb{C}^\ell$.

$$x \cdot \left( \sum_{i=1}^{\ell} w_i \otimes \varepsilon_i \right) = \sum_{i=1}^{\ell} \pi \left( g_{\sigma(x)(i)}^{-1} x g_i \right) (w_i) \otimes \varepsilon_{\sigma(x)(i)}$$

Verify that the above equation defines a representation of $G$ on $W \otimes \mathbb{C}^\ell$. Prove that $\mathrm{Ind}_H^G(W) \xrightarrow{\sim} W \otimes \mathbb{C}^\ell$ as representations of $G$.

**Problem 2.** Let $V$ be the standard irreducible 2–dimensional representation of $S_3$ and $\mathbb{I}$ be the 1–dimensional trivial representation of $S_2 \times S_2$. Compute the dimension of

$$\mathrm{Hom}_G \left( \mathrm{Ind}_{S_2 \times S_2}^{S_4}(\mathbb{I}), \mathrm{Ind}_{S_3}^{S_4}(V) \right)$$

**Problem 3.** With the set up of the previous problem, let $W$ be the representation of $S_4$ defined by pull–back of $V$ under $S_4 \to S_4/(S_2 \times S_2) \xrightarrow{\sim} S_3$. Let $\widetilde{V}$ be the standard irreducible 3–dimensional representation of $S_4$. Prove that

$$\mathrm{Ind}_{S_2 \times S_2}^{S_4}(\mathbb{I}) = \mathbb{I} \oplus \widetilde{V} \oplus W$$

**Problem 4.** Let $G$ be a finite group and $f \in (\mathbb{C}G)_{\mathrm{class}}$. Prove that $f$ is a character of an irreducible finite–dimensional representation of $G$ if, and only if the following three conditions are satisfied:

- $f$ is a $\mathbb{Z}$–linear combination of the characters of some finite–dimensional representations of $G$.
- $(f, f) = 1$.
- $f(e) > 0$ where $e$ is the identity element of $G$.

**Problem 5.** For each partition $\lambda$ of 5, let $\iota_\lambda \in (\mathbb{C}S_5)_{\mathrm{class}}$ be the character of the induced representation $\mathrm{Ind}_{S_\lambda}^{S_5}(\mathbb{I})$. Prove that the following class function is the character of some finite–dimensional irreducible representation of $S_5$:

$$s_{(3,2)} := \iota_{(3,2)} - \iota_{(4,1)}$$