ALGEBRA 1: PROBLEM SET 10

$A =$ a commutative ring in all the problems below.

Problem 1. *[Nilradical of a ring]* Let $\mathfrak{n} \subset A$ consist of all nilpotent elements. Prove that

$$\mathfrak{n} = \bigcap_{\substack{\mathfrak{p} \subset A \\ \text{prime ideal}}} \mathfrak{p}$$

Problem 2. Let $\mathfrak{a} \subset A$ be the set of all zero–divisors of A. Is $\mathfrak{a}$ an ideal of A?

Problem 3. Let $n \in A$ be a nilpotent element and $u \in A$ be a unit (that is, u has a multiplicative inverse). Prove that $u + n$ is again a unit.

Problem 4. Let $S \subset A$ be a set such that $0 \notin S$ and $r, s \in S$ implies $rs \in S$ (multiplicatively closed set). Let $\mathfrak{p}$ be an ideal which is maximal among the ideals not intersecting S. That is, maximal with respect to inclusion, from the following set:

$$I_S = \{\mathfrak{a} \subset A \text{ an ideal such that } \mathfrak{a} \cap S = \emptyset\}$$

Prove that $\mathfrak{p}$ is prime.

Problem 5. If for every $x \in A$ there exists an $n \geq 2$ such that $x^n = x$, then prove that every prime ideal in A is maximal.

Problem 6. Let B be another commutative ring and let $f : A \to B$ be a ring homomorphism. Let $\mathfrak{p} \subset A$ be a prime ideal and define $\mathfrak{b} \subset B$ to be the ideal generated by $f(\mathfrak{p})$. Prove or disprove: $\mathfrak{b}$ is a prime ideal.

Problem 7. Let K be a field and $R = K[[x]]$ be the ring of formal power series in a variable x with coefficients from K. Prove that R is a local ring, with unique maximal ideal $\mathfrak{m} = (x)$.

Problem 8. Prove that the set of prime ideals in A has a minimal element with respect to inclusion.

Problem 9. *[Jacobson radical]* Consider the following subset of A:

$$\mathfrak{J} := \{x \in A \text{ such that for every } y \in A, 1 - xy \text{ is a unit}\}$$

Prove that $\mathfrak{J}$ is an ideal of A. Further prove that

$$\mathfrak{J} = \bigcap_{\substack{\mathfrak{m} \subset A \\ \text{maximal ideal}}} \mathfrak{m}$$

Problem 10. Consider the ring $R = \mathbb{C}[x]/(f)$ where $f \in \mathbb{C}[x]$ is a non–zero polynomial. Let us write $f(x) = \prod_{i=1}^{\ell}(x-a_i)^{n_i}$ where $a_1, \ldots, a_\ell \in \mathbb{C}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}_{\geq 1}$.

Prove that $R \xrightarrow{\sim} \prod_{i=1}^{\ell} \mathbb{C}[x]/(x-a_i)^{n_i}$.

Problem 11. Let $R := \mathbb{Z}[x]/(x^2+1)$.

(a) Prove or disprove: R is a principal ideal domain.

(b) Let $p \in \mathbb{Z}$ be a prime number such that $p \equiv 3(\bmod 4)$. Prove that the ideal generated by p in R is a prime ideal.

Problem 12. *[Radical of an ideal]* Given an ideal $\mathfrak{a} \subset A$, define $r(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a}\}$. Prove the following statements for two ideals $\mathfrak{a}, \mathfrak{b}$ in A.

(a) $r(\mathfrak{a})$ is an ideal containing $\mathfrak{a}$.

(b) $r(r(\mathfrak{a})) = r(\mathfrak{a})$.

(c) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

(d) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$.

Problem 13. Let $\mathfrak{p}$ be a prime ideal of A. Prove that $r(\mathfrak{p}^n) = \mathfrak{p}$ for every $n \geq 1$.

Problem 14. *[Ideal quotients]* Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Define

$$(\mathfrak{a} : \mathfrak{b}) := \{a \in A \text{ such that } a\mathfrak{b} \subset \mathfrak{a}\}$$

Prove that $(\mathfrak{a} : \mathfrak{b})$ is an ideal in A.

Problem 15. With the definition given in the previous problem, take A to be $\mathbb{Z}$. Compute the ideal quotient $((n) : (m))$ for two positive integeres $n, m \in \mathbb{Z}$.