ALGEBRA 1: PROBLEM SET 11

$A =$ a commutative ring in all the problems below.

Problem 1. Let $\mathfrak{m} \subsetneq A$ be a maximal ideal, and let $\mathfrak{p} \subsetneq A$ be a prime ideal. Assume there exists $n \geq 1$ such that $\mathfrak{m}^n \subset \mathfrak{p}$. Prove that $\mathfrak{m} = \mathfrak{p}$.

Problem 2. Assume $S \subset A$ is a subset closed under multiplication and addition. Let $\mathfrak{p}_i \subset A$ $(1 \leq i \leq n)$ be a finite set of ideals in A such that at most two of them are not prime. Prove that if $S \subset \cup_{i=1}^n \mathfrak{p}_i$ then there exists $j \in \{1, \ldots, n\}$ such that $S \subset \mathfrak{p}_j$.

Problem 3. Let $\mathfrak{a} \subset A$ be an ideal and M be an A–module. Let $\mathfrak{a}M := \{ax : a \in \mathfrak{a}, x \in M\}$ a submodule of M. Prove that $M/\mathfrak{a}M$ satisfies the universal property of $(A/\mathfrak{a}) \otimes_A M$, and hence the two are isomorphic.

Problem 4. Use the previous problem to show that if $\mathfrak{a}, \mathfrak{b} \subset A$ are two coprime ideals, then $(A/\mathfrak{a}) \otimes_A (A/\mathfrak{b}) = 0$ (the trivial A–module).

Problem 5. Let $\{M_i\}_{i \in I}$ be a set of A–modules and let N be another A–module. Prove that

$$\left(\bigoplus_{i \in I} M_i \right) \otimes_A N \xrightarrow{\sim} \bigoplus_{i \in I} (M_i \otimes_A N)$$

Problem 6. Let M be an A–module and assume we have $p \in \mathrm{End}_A(M)$ such that $p^2 = p$ (called an *idempotent*). Prove that $M \cong M_1 \oplus M_2$ where M_1 is the kernel of p and M_2 is the image of p.

Problem 7. Give a counterexample to the assertion of Problem 7, if we do not impose the condition $p^2 = p$.

Take $A = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$ in Problems 8-10 below.

Problem 8. Prove that $M \otimes_{\mathbb{Z}} N \xrightarrow{\sim} \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Problem 9. Given $\alpha \in M$, define P_α to be the abelian group generated by two elements e_1, e_2 subject to the following relations:

$$me_1 = 0; \qquad ne_2 = \alpha e_1$$

Verify that we have a natural short exact sequence:

$$\mathcal{E}_\alpha : \qquad 0 \to M \to P_\alpha \to N \to 0$$

Problem 10. Determine the necessary and sufficient conditions for two short exact sequences $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ to be isomorphic.