ALGEBRA 1: PROBLEM SET 12

$A =$ a commutative ring in all the problems below.
$S =$ a multiplicatively closed subset of A (recall $1 \in S$ and $0 \notin S$).
$j_S : A \to S^{-1}A$ the natural ring homomorphism.

Problem 1. Let M be an A–module. Prove that $S^{-1}A \otimes_A M \equiv S^{-1}M$.

Problem 2. Let $\mathfrak{a} \subset A$ be an ideal. Prove that $S^{-1}\mathfrak{a}$ is the ideal in $S^{-1}A$ generated by $j_S(\mathfrak{a})$.

Problem 3. Let B be another commutative ring which contains A as a subring. Assume that B is finitely generated (as a ring over A). Prove that, if A is Noetherian, then so is B.

Problem 4. Assume $A[x]$ is Noetherian. Does it imply that A is Noetherian?

Problem 5. Consider the following sequence of A–linear maps between A–modules.

$$0 \to M' \to M \to M'' \to 0$$

Prove that this sequence is exact if, and only if, for every maximal ideal $\mathfrak{m} \subsetneq A$, the following sequence of $A_{\mathfrak{m}}$–modules is exact.

$$0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}} \to 0$$

Problem 6. Assume that A is not Noetherian. Let $\mathcal{S}$ be the set of all ideals of A which are not finitely–generated. Prove that this set has a maximal element and any such maximal element is necessarily a prime ideal of A.

Problem 7. Assume that A is Noetherian. Prove that so is $A[[x]]$.

Problem 8. Assume that $A_{\mathfrak{p}}$ is a Noetherian ring, for every prime ideal $\mathfrak{p} \subsetneq A$. Prove or disprove: A is Noetherian.

Problem 9. Let M be a Noetherian module over A. Prove that $M[x]$ is a Noetherian module over $A[x]$.

Problem 10. Let M be a Noetherian module over A. Let $f : M \to M$ be a *surjective* A–linear map. Prove that f is an isomorphism.
Hint.– consider the chain of submodules $\mathrm{Ker}(u^n)$.