ALGEBRA 1: PROBLEM SET 13

In the problems below A denotes a commutative ring.

Problem 1. Let A be a Noetherian ring and $\mathfrak{a} \subset A$ be an ideal. Prove that there exists $n \geq 1$ such that $r(\mathfrak{a})^n \subset \mathfrak{a}$.

Problem 2. Let $\mathfrak{a} \subset A$ be an ideal and assume that $r(\mathfrak{a})$ is maximal. Prove that $\mathfrak{a}$ is primary.

Problem 3. Let $\mathfrak{m} \subset A$ be a maximal ideal. Prove that $\mathfrak{m}^n$ is primary for every $n \geq 1$.

Problem 4. Assume A is Noetherian and let $\mathfrak{p} \subset A$ be a prime ideal. Prove that $A_{\mathfrak{p}}$ is Artinian if, and only if $\mathfrak{p}$ is a minimal prime ideal.

Problem 5. Let $\mathfrak{p} \subset A$ be a prime ideal and consider the natural ring homomorphism $\varphi : A \to A_{\mathfrak{p}}$. For any $n \geq 1$, let $\mathfrak{q}_n$ be the ideal in $A_{\mathfrak{p}}$ generated by $\varphi(\mathfrak{p}^n)$. Prove that $\varphi^{-1}(\mathfrak{q}_n)$ is primary.

Problem 6. Assume that A is Noetherian and let $\mathfrak{a} \subset A$ be an ideal. Assume that $\mathfrak{a} = r(\mathfrak{a})$. Prove that $\mathrm{Min}(\mathfrak{a}) = \mathrm{Assoc}(\mathfrak{a})$.

Problem 7. Let $\mathfrak{a} \subset A$ be an ideal. Prove that

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p}$$

Problem 8. Consider the ideal $\mathfrak{m} = (2, t) \subset \mathbb{Z}[t]$ and $\mathfrak{a} = (4, t) \subset \mathbb{Z}[t]$. Prove that $\mathfrak{a}$ is primary and $r(\mathfrak{a}) = \mathfrak{m}$. Prove that $\mathfrak{a}$ is not equal to any power of $\mathfrak{m}$.

Problem 9. Let $\mathfrak{a} = (x^2, xy) \subset A = K[x, y]$ where K is any field. For any $n \geq 2$ define $\mathfrak{q}_n = (x^2, xy, y^n)$. Prove that

(1) $\mathfrak{q}_n$ is primary.

(2) $\mathfrak{a} = (x) \cap \mathfrak{q}_n$.

(3) $r(\mathfrak{q}_n) = (x, y)$.