Lecture 1

§1.0 Recall: last time we defined monoids, groups, homomorphisms, subgroups and normal subgroups. We proved that if $\varphi : G \longrightarrow G'$ is a homomorphism then

$$\mathrm{Ker}(\varphi) := \{x \in G : \varphi(x) = e'\} \subset G \text{ is a normal}$$

subgroup. Now we give a converse of this statement.

§1.1. Let G be a group and $H < G$ a subgroup.

Definition. Consider the relation $\sim$ on G given by

$$x \sim y \quad \text{if} \quad x^{-1}y \in H$$

Lemma. $\sim$ is an equivalence relation.

Proof Symmetry: $x \sim y \Rightarrow x^{-1}y \in H \Rightarrow (x^{-1}y)^{-1} \in H$ (as H is a subgroup)

$\Rightarrow y^{-1}x \in H \Rightarrow y \sim x.$

Reflexive: $x \sim x$ since $x^{-1}x = e \in H.$

Transitivity: $x \sim y$ and $y \sim z$

$\Rightarrow x^{-1}y,\ y^{-1}z \in H \Rightarrow (x^{-1}y)\cdot(y^{-1}z) \in H$

$\Rightarrow x^{-1}z \in H \quad \Rightarrow \quad x \sim z$ $\square$

G/H is defined to be the set of equivalence classes in G with respect to $\sim$ defined above.

§1.2. Remarks: $x^{-1}y \in H$ is equivalent to saying that $x.H = y.H$. The equivalence classes mentioned above are known as <u>left cosets</u> (modulo H). One can similarly define the notion of <u>right cosets</u>, and $H\backslash G$, by considering the equivalence relation $x \sim y$ if $yx^{-1} \in H$.

§1.3. Now assume H is normal in G.

Prop. G/H carries a group structure induced from the one on G (explicitly $(g_1H)\cdot(g_2H) := g_1g_2H$ $(gH)^{-1} := g^{-1}H$, identity = coset of $e = e.H$).
The natural projection $\pi : G \longrightarrow G/H$ is a group homomorphism and $\mathrm{Ker}(\pi) = H$.

Proof. First of all we need to prove that the law of composition $(g_1H)\cdot(g_2H) = g_1g_2H$ is well-defined.

i.e. $\begin{matrix} g_1 \sim g_1' \\ g_2 \sim g_2' \end{matrix} \Rightarrow g_1g_2 \sim g_1'g_2'$ (To prove).

$g_\ell \sim g_\ell'$ $(\ell = 1,2)$ $\Rightarrow$ $\begin{matrix} g_1^{-1}g_1' \\ g_2^{-1}g_2' \end{matrix} \in H$. Now $(g_1g_2)^{-1}g_1'g_2' =$

$$= g_2^{-1} g_1^{-1} g_1' g_2' = g_2^{-1} \underbrace{(g_1^{-1} g_1')}_{\in H} g_2 \underbrace{g_2^{-1} g_2'}_{\in H}$$

$g_2^{-1} (g_1^{-1} g_1') g_2 \in H$ <u>since H is normal</u>. So

$$(g_1 g_2)^{-1} (g_1' g_2') \in H \implies g_1 g_2 \sim g_1' g_2'.$$

Thus we have a well defined law of composition on G/H.

It is clearly associative: $g_1 H \cdot ((g_2 H) \cdot (g_3 H))$

$$= ((g_1 H) \cdot (g_2 H)) \cdot g_3 H = g_1 g_2 g_3 H.$$

Identity element: $e \cdot H$ (easy).

Inverse element: $(gH) \cdot (g^{-1} H) = g g^{-1} H = e \cdot H.$

$(g^{-1} H) \cdot (gH) = g^{-1} g H = e \cdot H$

$$\implies (gH)^{-1} = g^{-1} H.$$

The remaining assertions are obvious and left as an exercise $\square$

§1.4. Consider the abelian group $\mathbb{Z}, +$. We know every subgroup of an abelian group is automatically normal.

Lemma. Let $N \subset \mathbb{Z}$ be a subgroup. Then $\exists\, n \in \mathbb{Z}_{\geq 0}$ s.t.

$$N = n \cdot \mathbb{Z} = \{0, \pm n, \pm 2n, \dots\}$$

Proof. Assume $N \neq \{0\}$ (if $N = \{0\}$, then $N = 0 \cdot \mathbb{Z}$).

Let n be the smallest positive integer in N.

Claim. $N = n \cdot \mathbb{Z}$

Since N is a subgroup, $n \cdot \mathbb{Z} \subset N$. If $N \not\subset n \cdot \mathbb{Z}$, ($n$ has to $\neq 1$ in this case) then $\exists\, p \in N \setminus (n \cdot \mathbb{Z})$. We may assume $p > 0$. Let k be such that $kn < p < (k+1)n$. Then $p - kn \in N$ and $0 < p - kn < n$ contradicts minimality of n $\square$

We get more examples of abelian groups this way:

$\mathbb{Z}/n\mathbb{Z}$ with law of composition $=$ addition modulo n.

All these are examples of <u>cyclic groups</u>.

A group G is <u>cyclic</u> if $\exists\, a \in G$ such that every element of G is of the form a^m $(m \in \mathbb{Z})$.

$$a^m = \left[\underbrace{a \cdot \dots \cdot a}_{\substack{m\text{-times} \\ \text{if } m>0}} \;;\; \underbrace{a^{-1} \cdot \dots \cdot a^{-1}}_{\substack{-m \text{ times} \\ \text{if } m<0}}\right]$$

This element a is called generator of G (it is not unique; neither is m that shows up in the definition above).

e.g. $\{\pm 1\} \leftarrow$ set of generators of $\mathbb{Z}$.

Exercise: Number of generators of $\mathbb{Z}/n\mathbb{Z}$

$= \phi(n) = \#$ of $p \in \{1, \dots, n-1\}$ s.t. $\gcd(p, n) = 1$.

§1.5. Subgroups generated by a set.

Lemma. Let H_1, H_2 be two subgroups of a group G. Then so is $H_1 \cap H_2$. If H_1, H_2 are normal then so is $H_1 \cap H_2$.

Proof is easy and works for arbitrary intersections.

As a consequence of this, if $X \subset G$ is a subset we can define $\langle X \rangle \subset G$ to be the smallest subgroup containing X. (called <u>subgroup generated by X</u>).

Similarly we can consider the smallest normal subgroup containing X (I will denote it by $N\langle X\rangle$ ← not a standard notation).

§1.6. Groups in terms of generators and relations

In practice, groups arise (extrinsically) as symmetries of a mathematical structure. For example,

X = a metric space $\quad G = \{X \xrightarrow{f} X \text{ isometry}\}$.

X = a smooth manifold $\quad G = \{X \xrightarrow{f} X \text{ diffeomorphism}\}$

More intrinsically we can get a group in terms of generators and relations.

Let A be a set.

Definition. The free group on A (or generated by A), $F(A)$ consists of sequences

$$(a_1, n_1)\cdot(a_2, n_2)\cdot \ldots (a_\ell, n_\ell) \quad \text{where}$$

$a_1, \ldots, a_\ell \in A \qquad a_j \neq a_{j+1} \quad (1 \leq j \leq \ell - 1)$

$n_1, \ldots, n_\ell \in \mathbb{Z}$

Law of composition on $\mathcal{F}(A)$ = concatenation of sequences, with the following rule of cancellation

$$(a, n)\,(a, m) = (a, n+m).$$

ϵ = empty sequence is the identity element.

$$\left[(a_1, n_1) \cdots (a_\ell, n_\ell)\right]^{-1} = (a_\ell, -n_\ell) \cdots (a_1, -n_1) \quad \text{[inverse]}$$

Definition. Let A be a set and $R \subset \mathcal{F}(A)$

A group presented by <u>generating set A</u> and <u>relations R</u>, denoted by $G = \langle A ; R \rangle$ is defined as

$$G = \mathcal{F}(A) \big/ N\langle R \rangle$$

smallest normal subgroup containing R.

§1.7 Examples

(i) $G = \langle x, y \,;\, x^2, y^2, xyxyxy \rangle$

$= \{e, x, y, xy, yx, xyx = yxy\}$

$\simeq S_3$ (symmetric group on 3 letters)

via isomorphism
$$\begin{aligned} G &\longrightarrow S_3 \\ x &\longmapsto (12) \\ y &\longmapsto (23) \end{aligned}$$

Exercise: Verify that the map given above is an isomorphism of groups.

(ii) $G = \langle x, y \,;\, xy^2 = y^3 x \text{ and } yx^2 = x^3 y \rangle$

Exercise (we did this in class)
Prove that $G = \{e\}$: trivial group

(iii) $A = \{x, y\}$ $\quad G = F(A)$ free group on 2 letters.

$$\begin{aligned} \varphi : G &\longrightarrow \mathbb{Z} \times \mathbb{Z} \\ w &\longmapsto (\#\text{ of } x\text{'s in } w, \#\text{ of } y\text{'s in } w) \end{aligned}$$

Ex. Check that φ is a group homomorphism

Fun activity: read ~~the~~ a proof of: $N = \mathrm{Ker}(\varphi)$ is NOT finitely generated; i.e. $\forall \{w_1, \dots, w_n\}$ finite subset of N,
$$\langle \{w_1, \dots, w_n\} \rangle \subsetneq N.$$