Lecture 3

(3.0) Recall: last time we introduced the notion of a group G acting on a set X; denoted by $G \circlearrowright X$.

We proved two counting lemmas:

$$|X| = \sum_{\alpha \in G\backslash X} \frac{|G|}{|\mathrm{Stab}(x_\alpha)|}$$

$$|G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

recall $\mathrm{Stab}(x) = \{g \in G : g.x = x\}$ subgroup of G.

$X^g = \{x \in X : g.x = x\}$ subset of X

(3.1) Some examples: let G be a group. Take $X = G$.

- Left multiplication ~~#~~ $L : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(G)$
  $g \longmapsto L_g$

$$L_g(x) = gx$$

- Right multiplication $R : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(G)$

$$R_g(x) = xg^{-1}$$

- Conjugation (problem #12 of Problem Set 1)

$$C : G \longrightarrow \mathrm{Aut}_{\mathrm{Group}}(G) \quad \text{by :}$$
$$g \longmapsto C_g$$

$$C_g(x) = g x g^{-1}$$

(3.2) For now consider $G \circlearrowright G$ by conjugation:

For $x \in G$, the stabilizer $= \{ g \in G : g x g^{-1} = x \}$
is also called <u>centralizer of x</u>, denoted by $Z_G(x)$.

G-orbits under this action are called <u>conjugacy classes</u>

Check: for $g \in G$, the set of elements of G fixed under g is $Z_G(g)$.

Our counting lemmas redd:

- $$|G| = \sum_{\alpha \in \mathcal{C}} (G : Z_G(x_\alpha)) \quad \text{where}$$

$\mathcal{C}$ = set of conjugacy classes. $\left\{ \begin{array}{l} \text{As before, } x_\alpha \text{ is a} \\ \text{representative from } \alpha \end{array} \right\}$

- $$\underset{\text{\# of conjugacy classes}}{\underset{\uparrow}{|\mathcal{C}|}} = \underbrace{\frac{1}{|G|}\sum_{g\in G}|Z_G(g)|}_{\text{average \# of elements in a centralizer.}}$$

(3.3) This week we will prove some isomorphism theorems. The most fundamental one is

Theorem. Let G and G' be two groups and $\varphi : G \to G'$ be a group hom. Let $K = \mathrm{Ker}(\varphi)$ be the kernel of φ and $H' = \mathrm{Im}(\varphi)$ be its image. Then we have

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\ G/K \arrow[r, "\overline{\varphi}"'] & H' \arrow[u, "i"'] \end{tikzcd}$$

commutative diagram

Here π is the natural projection and i is the inclusion. Moreover $\overline{\varphi}$ is an isomorphism.

Proof. Define $\psi : G/K \longrightarrow G'$ by

$\psi(gK) = \varphi(g)$. We need to make sure it is well-defined, i.e. [if $g_1K = g_2K$ then $\varphi(g_1) = \varphi(g_2)$]

To prove

$$g_1K = g_2K \iff g_1^{-1}g_2 \in K \overset{\text{by defn. of } \operatorname{Ker}(\varphi)}{\iff} \varphi(g_1^{-1}g_2) = e' \quad \text{(identity of } G'\text{)}$$

i.e. $\varphi(g_1)^{-1}\varphi(g_2) = e'$

$\iff \varphi(g_1) = \varphi(g_2)$

- ψ is a group hom. : $\psi(g_1K \cdot g_2K) \overset{\text{by defn. of law of composition on } G/K}{=} \psi(g_1g_2K)$

$$= \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \psi(g_1K)\psi(g_2K)$$

- ψ is injective : $\psi(gK) = e' \iff \varphi(g) = e'$

$\iff g \in K$ i.e. $gK = K$.

- $\bar{\varphi} = \psi$ with range restricted to $H' = \operatorname{Im}(\varphi)$

By definition $\bar{\varphi}$ is surjective and hence an isomorphism $\square$

(3.4) Cor. Let $\varphi : G \longrightarrow G'$ be a surjective hom. of groups. Then $G' \simeq G/\mathrm{Ker}(\varphi)$.

The statement of this corollary is often written as:

We have an <u>exact sequence</u> (see definition below)

$$\mathbb{1} \longrightarrow \mathrm{Ker}(\varphi) \xrightarrow{i} G \xrightarrow{\varphi} G' \longrightarrow \mathbb{1}$$

where
- $\mathbb{1} = \{1\}$ is the trivial group.
- $i : \mathrm{Ker}(\varphi) \longrightarrow G$ is the natural inclusion
- $\mathbb{1} \longrightarrow \mathrm{Ker}(\varphi) \quad : \quad 1 \longmapsto e$

  $G' \longrightarrow \mathbb{1} \quad : \quad g' \longmapsto 1 \quad \forall g' \in G'$

(3.5) Definition: A sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3$$ is said to be <u>exact</u>

(or exact at G_2) if $\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$.

- $\mathbb{1} \longrightarrow G_1 \xrightarrow{\varphi} G_2$ exact $\Longleftrightarrow$ φ is injective
- $G_2 \xrightarrow{\psi} G_3 \longrightarrow \mathbb{1}$ exact $\Longleftrightarrow$ ψ is surjective.

An exact sequence of the form

$$\mathbb{1} \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow \mathbb{1}$$

is usually referred to as a <u>short exact sequence</u>. It signifies that (i) G_1 can be viewed as a normal subgroup of G_2

$$G_1 \xrightarrow{\sim} \mathrm{Im}(\varphi) \subset G_2$$

(ii) $$G_2/\mathrm{Im}(\varphi) = G_2/\mathrm{Ker}(\psi) \xrightarrow[\psi]{\sim} G_3 .$$

Ex. Rephrase the statement of Thm (3.3) page 3, in terms of short exact sequences.

$$G \xrightarrow{\varphi} G' \quad \rightsquigarrow \quad \begin{array}{ccccccccc} \mathbb{1} & \rightarrow & \mathrm{Ker}(\varphi) & \rightarrow & G & \rightarrow & \mathrm{Im}(\varphi) & \rightarrow & \mathbb{1} \\ & & \| & & \| & & \uparrow \simeq & & \\ \mathbb{1} & \rightarrow & \mathrm{Ker}(\varphi) & \rightarrow & G & \rightarrow & G/\mathrm{Ker}(\varphi) & \rightarrow & \mathbb{1} \end{array}$$

The exact sequence $\mathbb{1} \rightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \rightarrow \mathbb{1}$ also signifies that "G_2 is built out of G_1 and G_3".

More accurately G_2 is an extension of G_3 by G_1 (to be defined later).

(3.6) Some examples of short exact sequences

(1) For abelian groups, the trivial group $\mathbb{1}$ is often just written as 0.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}/5\mathbb{Z} & \to & 0 \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & \\ & & m & \mapsto & 5\cdot m & & & & \end{array}$$

is a short exact sequence.

(2)
$$\det : GL_2(\mathbb{C}) \longrightarrow \mathbb{C} \setminus \{0\} =: \mathbb{C}^\times$$
$$A \longmapsto \det(A)$$

(group under multiplication)

is a surjective group hom.

$\mathrm{Ker}(\det) = 2\times 2$ matrices of determinant 1 $(=: SL_2(\mathbb{C}))$

$$\mathbb{1} \longrightarrow SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^\times \to \mathbb{1}$$

is a short exact sequence

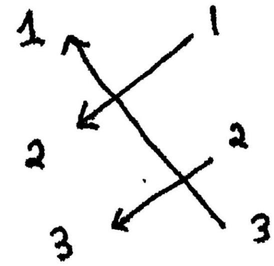
(3) $S_n \ni \sigma$. Define $\ell(\sigma) = \#\{i<j : \sigma(i) > \sigma(j)\}$

We will prove later that

$$\mathrm{sign} : S_n \longrightarrow \{\pm 1\}$$
$$\sigma \longmapsto (-1)^{\ell(\sigma)}$$

is a group hom.

e.g. $\sigma = (1\,2\,3) \in S_3$ can be viewed pictorially as (read right to left)

$\ell(\sigma)$ = # of crossings in 

$= 2$

$\mathrm{sign}(\sigma) = +1$.

$A_n := \mathrm{Ker}(\mathrm{sign})$ subgroup of even permutations.

$$\mathbb{1} \longrightarrow A_n \longrightarrow S_n \longrightarrow \{\pm 1\} \longrightarrow \mathbb{1}$$

is an exact sequence.

(3.7) Definition: A group G is called <u>simple</u> if it does not contain any proper, non-trivial, normal subgroups

[i.e., $N \triangleleft G \Rightarrow N = \{e\}$ or $N = G$]

We will see a proof later that A_n is ~~no~~ simple for $n \geq 5$.