

Lecture 4

①

(4.0) Recall : last time we proved that for each group hom.

$\varphi: G \rightarrow G'$ we get an isomorphism

$$\bar{\varphi}: G/\text{Ker}(\varphi) \xrightarrow{\sim} \text{Im}(\varphi) \quad \text{making the following}$$

diagram commute

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \bar{\pi} \downarrow & & \uparrow i \\ G/\text{Ker}(\varphi) & \xrightarrow[\bar{\varphi}]{\sim} & \text{Im}(\varphi) \end{array}$$

(4.1) Let G be a group and $N \triangleleft G$ a normal subgroup.

Theorem. (i) The assignment $H \mapsto H/N$ is a

bijection between $\left\{ \begin{array}{l} \text{Subgroups of} \\ G \text{ containing } N \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Subgroups of} \\ G/N \end{array} \right\}$

(ii) Let $H < G$ be a subgroup containing N . Then H is normal if and only if $H/N < G/N$ is.

In this case we have an isomorphism

$$G/H \xrightarrow{\sim} (G/N)/(H/N)$$

Proof. Let $\pi : G \longrightarrow G/N$ be the natural (2)

surjection. If $H < G$ is a subgroup of G containing N ,

then $\pi(H) = \{hN : h \in H\}$ is a subgroup of G/N :

- $e_{G/N} = e \cdot N \in \pi(H)$ (identity of G/N)
- by the law of composition of G/N , $(h_1N) \cdot (h_2N) = h_1 h_2 N \in \pi(H) \quad \forall h_1, h_2 \in H$

• $(hN)^{-1} = h^{-1}N \in \pi(H) \quad \forall h \in H.$

[Note: for any group hom $\varphi : G_1 \rightarrow G_2$ and a subgroup $H_1 < G_1$, $\varphi(H_1)$ is a subgroup of G_2]

Conversely if $\bar{H} < G/N$ is a subgroup, let $H = \pi^{-1}(\bar{H})$.

Claim: $H < G$ is a subgroup containing N and $\pi(H) = \bar{H}$.

Proof of the claim. $N = \pi^{-1}(\{e_{G/N}\}) \subset \pi^{-1}(\bar{H}) = H$.

Note $H = \{g \in G : \pi(g) \in \bar{H}\}$

• $e \in H$ is clear since $e \in N \subset H$

• $g_1, g_2 \in H \Rightarrow \pi(g_1 g_2) = \pi(g_1) \pi(g_2) \in \bar{H} \Rightarrow g_1 \cdot g_2 \in H$.

• $g \in H \Rightarrow \pi(g^{-1}) = \pi(g)^{-1} \in \bar{H} \Rightarrow g^{-1} \in H$.

$\pi(H) = \bar{H}$ by surjectivity of π .

□
(end of the proof of claim).

Now assume $H < G$ is a subgroup containing N , and let $\bar{H} = \pi(H) = H/N$.

H is normal $\Leftrightarrow \forall g \in G, h \in H$ we have $ghg^{-1} \in H$

$\Leftrightarrow ghg^{-1} \cdot N \in \bar{H} \Leftrightarrow (g \cdot N)(h \cdot N)(g \cdot N)^{-1} \in \bar{H}$

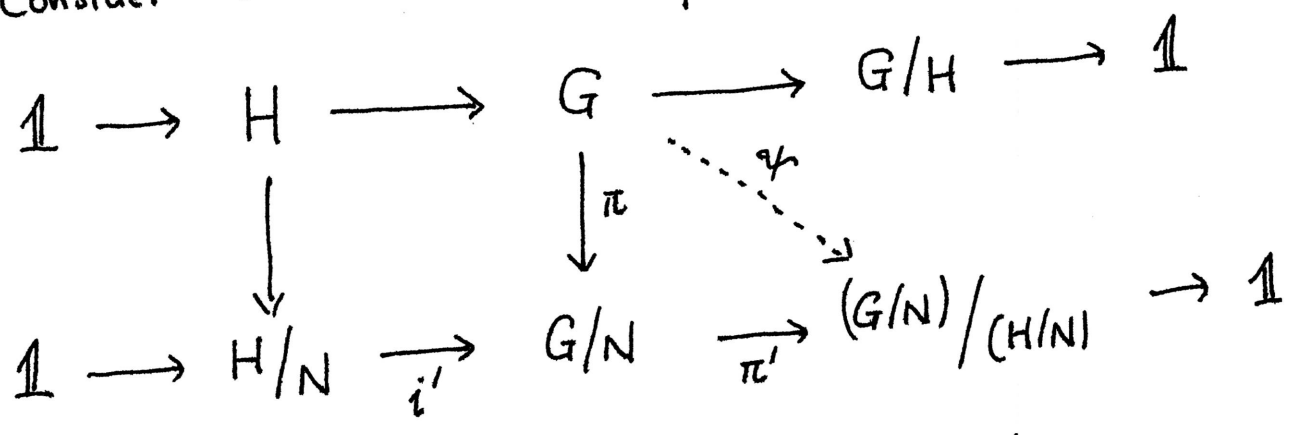
$\forall g \cdot N \in G/N$
 $h \cdot N \in H/N$

$\Leftrightarrow \bar{H} < \bar{G}$ is normal

[Note: for any group hom $\varphi: G_1 \rightarrow G_2$ and $N_2 \triangleleft G_2$, normal subgroup, $\varphi^{-1}(N_2) \triangleleft G_1$ is a normal subgroup.]

~~If φ is surjective~~

Consider the short exact sequences



- ψ is surjective since both π and π' are.
- $\text{Ker}(\psi) = H : \psi(g) = e \Leftrightarrow \pi'(\pi(g)) = e$
 $\Leftrightarrow \pi(g) \in \text{Ker}(\pi') = \text{Im}(i') = H/N$
 $\Leftrightarrow g \in H (= \pi^{-1}(H/N))$

So by Corollary to Theorem (3.3) (page 5), we get an iso. $G/H \longrightarrow (G/N)/(H/N)$ (4)

□

(4.2) Let G be a group, $H < G$ a subgroup and $N < G$ a normal subgroup.

(a) $H \cap N < H$ is a normal subgroup

(Proof. $\forall h \in H, x \in H \cap N$, we have

$hxh^{-1} \in H$ as H is a subgroup

$hxh^{-1} \in N$ as N is normal in G .

$\Rightarrow hxh^{-1} \in H \cap N$ □

(b) $H \cdot N (= \{h \cdot x : h \in H, x \in N\}) < G$. Then

$H \cdot N = N \cdot H$ is a subgroup of G

(Proof $h \cdot x = (h \cdot x \cdot h^{-1}) \cdot h \in N \cdot H \Rightarrow H \cdot N \subset N \cdot H$
similarly $N \cdot H \subset H \cdot N$.

$e = e \cdot e \in H \cdot N$

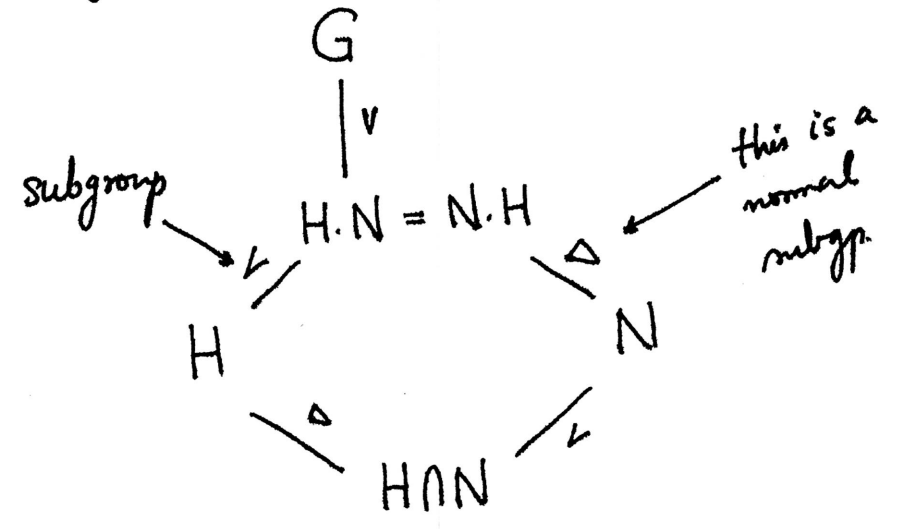
$(h_1 \cdot x_1) \cdot (h_2 \cdot x_2) = (h_1 \cdot h_2) \cdot (h_2^{-1} \cdot x_1 \cdot h_2 \cdot x_2) \in H \cdot N$

$(hx)^{-1} = x^{-1} \cdot h^{-1} \in N \cdot H = H \cdot N$ □

(c) $N < H \cdot N$ is a normal subgroup.

A small cartoon encoding all this data.

$H < G$
 $N \triangleleft G$

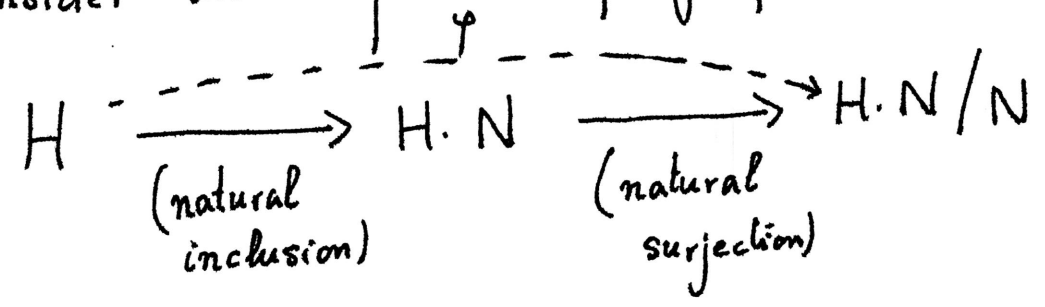


(4.3) Theorem.

$$\begin{array}{ccc} H / H \cap N & \longrightarrow & H \cdot N / N \\ (h \cdot (H \cap N)) & \longmapsto & hN \end{array}$$

is an isomorphism

Proof Consider the composition of group homs.



$H \xrightarrow{\varphi} H \cdot N / N$ is surjective $(h \cdot x \cdot N = h \cdot N = \varphi(h) \forall h \in H, x \in N)$

$\text{Ker}(\varphi) = H \cap N$: let $h \in H$. Then $\varphi(h) = \bar{e}$ (identity of $H \cdot N / N$)

if and only if $h \in N$ ($\Leftrightarrow h \in H \cap N$).

⑥

Again by Corollary (3.4) page 5, we get an iso.

$$H / H \cap N \longrightarrow H \cdot N / N$$

□

(4.4) Definition. $G = H \rtimes N$ (read: G is a semi-direct product of H and N) if

- (1) $H < G$ is a subgroup of G
- $N \triangleleft G$ is a normal subgroup of G
- (2) $H \cdot N (= N \cdot H \text{ see } \S 4.2 (b) \text{ page 4}) = G$.
- (3) $H \cap N = \{e\}$

G is a direct product of H and N , written as

$G = H \times N$ if in addition to (1) - (3) we have

$$hx = xh \quad \forall h \in H, x \in N$$

(4.5) Assume G is a group, $H < G$ a subgroup
 $N \triangleleft G$ a normal subgp.

Prop. ~~iff~~ $G = H \rtimes N$ iff we have

- a short exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \mathbb{1}$$

- a group hom. $s: H \longrightarrow G$ s.t. $\psi \circ s = \text{Id}_H$
(called a section)

[Short exact sequences for which a section exists, as above, are called split s.e.s.]

Proof. (\Rightarrow) If $G = H \rtimes N$, then by Theorem (4.3) of page 5,

$$H / H \cap N \xrightarrow{\sim} H \cdot N / N$$

but by defn $H \cap N = \{e\}$ and $H \cdot N = G$. Hence

$j: H \xrightarrow{\sim} G/N$ and thus we get a short exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \mathbb{1}$$

- φ is the natural inclusion

- ψ is the composition $G \longrightarrow G/N \xrightarrow{j^{-1}} H$

($j: H \xrightarrow{\sim} G/N$ is given by $j(h) = h \cdot N$)

Let $s: H \xrightarrow{\text{inclusion}} G$. Then $\psi \circ s = \text{Id}_H$ by definition of ψ .

(\Leftarrow) Given a split s.e.s.

$$\mathbb{1} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \mathbb{1}$$

$$\psi \circ s = \text{Id}_H$$

We view H (and N) as a subgroup (resp. normal subgroup) of G via s (and φ).

(in particular, s is injective)

• $G = H \cdot N$. Let $g \in G$. Define $h = s(\psi(g)) \in H$

$$\begin{aligned} \text{Then } \psi(h^{-1}g) &= \psi(s(\psi(g^{-1}))) \cdot \psi(g) \\ &= \psi(g)^{-1} \cdot \psi(g) \quad (\text{as } \psi \circ s = \text{Id}_H) \\ &= e_H \quad (\text{identity of } H) \end{aligned}$$

$$\Rightarrow h^{-1}g \in \text{Ker}(\psi) = \text{Im}(\varphi) \quad \text{identified with } N$$

$$\Rightarrow g \in H \cdot N.$$

• $H \cap N = \{e\}$: If $g = \varphi(x) = s(h)$ for some $x \in N, h \in H$

$$\begin{aligned} \text{then } \psi(g) &= \psi(\varphi(x)) = e_H \quad \text{as the sequence is exact} \\ &= \psi(s(h)) = h \quad (\text{as } \psi \circ s = \text{Id}) \end{aligned}$$

$$\Rightarrow h = e_H \Rightarrow g = e.$$

□