Lecture 4

(4.0) Recall : last time we proved that for each group hom. $\varphi : G \to G'$ we get an isomorphism

$$\overline{\varphi} : G/\mathrm{Ker}(\varphi) \xrightarrow{\sim} \mathrm{Im}(\varphi)$$

making the following diagram commute

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\
G/\mathrm{Ker}(\varphi) \arrow[r, "\sim", "\overline{\varphi}"'] & \mathrm{Im}(\varphi) \arrow[u, "i"']
\end{tikzcd}$$

(4.1) Let G be a group and $N \triangleleft G$ a normal subgroup.

Theorem. (i) The assignment $H \longmapsto H/N$ is a bijection between

$$\left\{ \begin{array}{l} \text{Subgroups of} \\ G \text{ containing } N \end{array} \right\} \to \left\{ \begin{array}{c} \text{Subgroups of} \\ G/N \end{array} \right\}$$

(ii) Let $H < G$ be a subgroup containing N. Then H is normal if and only if $H/N < G/N$ is.

In this case we have an isomorphism

$$G/H \xrightarrow{\sim} (G/N)/(H/N)$$

Proof. Let $\pi : G \longrightarrow G/N$ be the natural surjection. If $H < G$ is a subgroup of G containing N, then $\pi(H) = \{hN : h \in H\}$ is a subgroup of G/N :

- $e_{G/N} = e.N \in \pi(H)$
  (identity of G/N)
- by the law of composition of G/N. $(h_1N)\cdot(h_2N) = \underset{\in H}{h_1h_2}N$
  $\forall\, h_1, h_2 \in H$
- $(hN)^{-1} = h^{-1}\cdot N \in H \quad \forall\, h \in H.$

[Note: for any group hom $\varphi : G_1 \to G_2$ and a subgroup $H_1 < G_1$, $\varphi(H_1)$ is a subgroup of G_2]

Conversely if $\overline{H} < G/N$ is a subgroup, let $H = \pi^{-1}(\overline{H})$.

Claim: $H < G$ is a subgroup containing N and $\pi(H) = \overline{H}$.

Proof of the claim. $N = \pi^{-1}(\{e_{G/N}\}) \subset \pi^{-1}(\overline{H}) = H.$

Note $H = \{ g \in G : \pi(g) \in \overline{H} \}$

- $e \in H$ is clear since $e \in N \subset H$
- $g_1, g_2 \in H \Rightarrow \pi(g_1g_2) = \pi(g_1)\,\pi(g_2) \in \overline{H} \Rightarrow g_1 \cdot g_2 \in H.$
- $g \in H \Rightarrow \pi(g^{-1}) = \pi(g)^{-1} \in \overline{H} \Rightarrow g^{-1} \in H.$

$\pi(H) = \overline{H}$ by surjectivity of π. $\square$ (end of the proof of claim).

Now assume $H < G$ is a subgroup containing N, and let $\overline{H} = \pi(H) = H/N$.

H is normal $\iff$ $\forall\, g \in G,\ h \in H$ we have $ghg^{-1} \in H$

$\iff ghg^{-1}\cdot N \in \overline{H} \iff (g\cdot N)(h\cdot N)(g\cdot N)^{-1} \in \overline{H}$

$\forall\ g\cdot N \in G/N$
$h\cdot N \in H/N$

$\iff \overline{H} < \overline{G}$ is normal

[Note: for any group hom $\varphi: G_1 \to G_2$ and $N_2 \triangleleft G_2$, normal subgroup, $\varphi^{-1}(N_2) \triangleleft G_1$ is a normal subgroup.]

~~If φ is surjective~~

Consider the short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\pi"] \arrow[dr, dashed, "\psi"] & G/H \arrow[r] & 1 \\
1 \arrow[r] & H/N \arrow[r, "i'"'] & G/N \arrow[r, "\pi'"'] & (G/N)/(H/N) \arrow[r] & 1
\end{tikzcd}$$

- ψ is surjective since both π and π' are.
- $\mathrm{Ker}(\psi) = H$: $\psi(g) = e \iff \pi'(\pi(g)) = e$

$\iff \pi(g) \in \mathrm{Ker}(\pi') = \mathrm{Im}(i') = H/N$

$\iff g \in H \ (= \pi^{-1}(H/N))$.

So by Corollary to Theorem (3.3) (page 5), we get an iso. $G/H \longrightarrow (G/N)/(H/N)$

$\square$

(4.2) Let G be a group, $H < G$ a subgroup and $N \triangleleft G$ a normal subgroup.

(a) $H \cap N \subset H$ is a normal subgroup

(Proof. $\forall\, h \in H$, $x \in H \cap N$, we have

$hxh^{-1} \in H$ as H is a subgroup $\Rightarrow hxh^{-1} \in H \cap N$)

$hxh^{-1} \in N$ as N is normal in G. $\square$

(b) $H \cdot N \; (= \{h \cdot x : h \in H,\ x \in N\}) \subset G$. Then

$H \cdot N = N \cdot H$ is a subgroup of G

(Proof $\quad h \cdot x = (h \cdot x \cdot h^{-1}) \cdot h \in N \cdot H \Rightarrow H \cdot N \subset N \cdot H$

similarly $N \cdot H \subset H \cdot N$.

$e = e \cdot e \in H \cdot N$

$(h_1 x_1) \cdot (h_2 \cdot x_2) = (h_1 h_2) \cdot (h_2^{-1} x_1 h_2 x_2) \in H \cdot N$

$(hx)^{-1} = x^{-1} h^{-1} \in N \cdot H = H \cdot N$ $\square$)

(c) $N \subset H \cdot N$ is a normal subgp.

A small cartoon encoding all this data.

$H < G$

$N \lhd G$ $\rightsquigarrow$

subgroup

this is a normal subgp.

$$\begin{tikzcd}
 & G \arrow[d, phantom, "\vee"] & \\
 & H \cdot N = N \cdot H \arrow[dl, phantom, "\vee"] \arrow[dr, phantom, "\rhd"] & \\
H \arrow[dr, phantom, "\rhd"] & & N \arrow[dl, phantom, "\vee"] \\
 & H \cap N &
\end{tikzcd}$$

(4.3) Theorem.

$$H/_{H \cap N} \longrightarrow H \cdot N/_N$$

$$\left(h \cdot (H \cap N) \longmapsto h N \right)$$

is an isomorphism

Proof Consider the composition of group homs.

$$\begin{tikzcd}
H \arrow[r, "\text{(natural inclusion)}"'] \arrow[rr, dashed, bend left, "\varphi"] & H \cdot N \arrow[r, "\text{(natural surjection)}"'] & H \cdot N / N
\end{tikzcd}$$

$H \xrightarrow{\varphi} H \cdot N/_N$ is surjective $\big(h \cdot x \cdot N = h \cdot N$
$= \varphi(h)$
$\forall\, h \in H,\ x \in N \big)$

$Ker(\varphi) = H \cap N$: let $h \in H$. Then $\varphi(h) = \bar{e}$ (identity of $H \cdot N/_N$)

if and only if $h \in N$ ($\Leftrightarrow$ $h \in H \cap N$).

Again by Corollary (3.4) page 5, we get an iso.

$$H/_{H \cap N} \longrightarrow H \cdot N/_{N}$$

□

(4.4) Definition. $G = H \ltimes N$ (read: G is a semi-direct product of H and N) if

(1) $H < G$ is a subgroup of G
$N \triangleleft G$ is a normal subgroup of G

(2) $H \cdot N$ ($= N \cdot H$ see §4.2 (b) page 4) $= G$.

(3) $H \cap N = \{e\}$

G is a direct product of H and N, written as $G = H \times N$ if in addition to (1) – (3) we have

$$hx = xh \quad \forall\, h \in H,\ x \in N$$

(4.5) Assume G is a group, $H < G$ a subgroup, $N \triangleleft G$ a normal subgp.

Prop. ~~##~~ $G = H \ltimes N$ iff we have

- a short exact sequence

$$1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

- a group hom. $s : H \longrightarrow G$ s.t. $\psi \circ s = \mathrm{Id}_H$

(called a <u>section</u>)

[Short exact sequences for which a section exists, as above, are called <u>split</u> s.e.s.]

Proof. ($\Rightarrow$) If $G = H \ltimes N$, then by Theorem (4.3) of page 5, $H/_{H \cap N} \xrightarrow{\sim} H.N/_N$

but by defn $H \cap N = \{e\}$ and $H.N = G$. Hence

$j : H \xrightarrow{\sim} G/N$ and thus we get a short exact sequence

$$1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

- φ is the natural inclusion
- ψ is the composition $G \longrightarrow G/N \xrightarrow[j^{-1}]{\sim} H$

($j : H \xrightarrow{\sim} G/N$ is given by $j(h) = h.N$)

Let $s : H \underset{\text{inclusion}}{\hookrightarrow} G$. Then $\psi \circ s = \mathrm{Id}_H$ by definition of ψ. ∎

($\Leftarrow$) Given a split s.e.s.

$$\begin{tikzcd} \mathbb{1} \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] & H \arrow[r] \arrow[l, bend left, dashed, "s"] & \mathbb{1} \end{tikzcd}$$

$\psi \circ s = \mathrm{Id}_H$

(in particular, s is injective)

We view H (and N) as a subgroup (resp. normal subgp.) of G via s (and φ).

- $G = H \cdot N$. Let $g \in G$. Define $h = s(\psi(g)) \in H$

Then $\psi(h^{-1} g) = \psi(s(\psi(g^{-1}))) . \psi(g)$

$= \psi(g)^{-1} \cdot \psi(g)$ (as $\psi \circ s = \mathrm{Id}_H$)

$= e_H$ (identity of H)

$\Rightarrow h^{-1} g \in \mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$ identified with N

$\Rightarrow g \in H \cdot N$.

- $H \cap N = \{e\}$: If $g = \varphi(x) = s(h)$ for some $x \in N$, $h \in H$

then $\psi(g) = \psi(\varphi(x)) = e_H$ as the sequence is exact

$= \psi(s(h)) = h$ (as $\psi \circ s = \mathrm{Id}$)

$\Rightarrow h = e_H \Rightarrow g = e$. $\square$