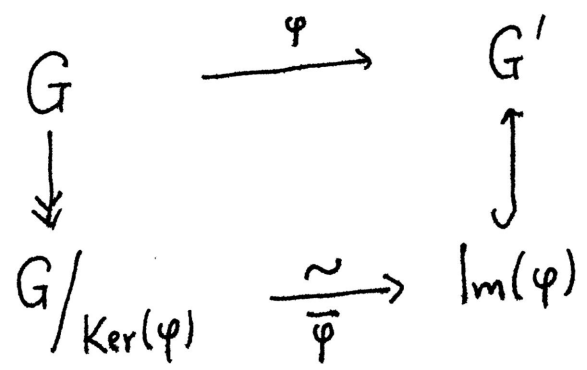


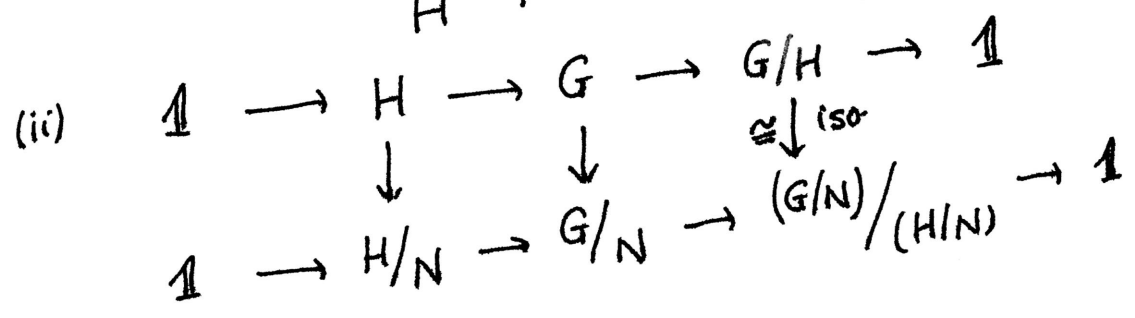
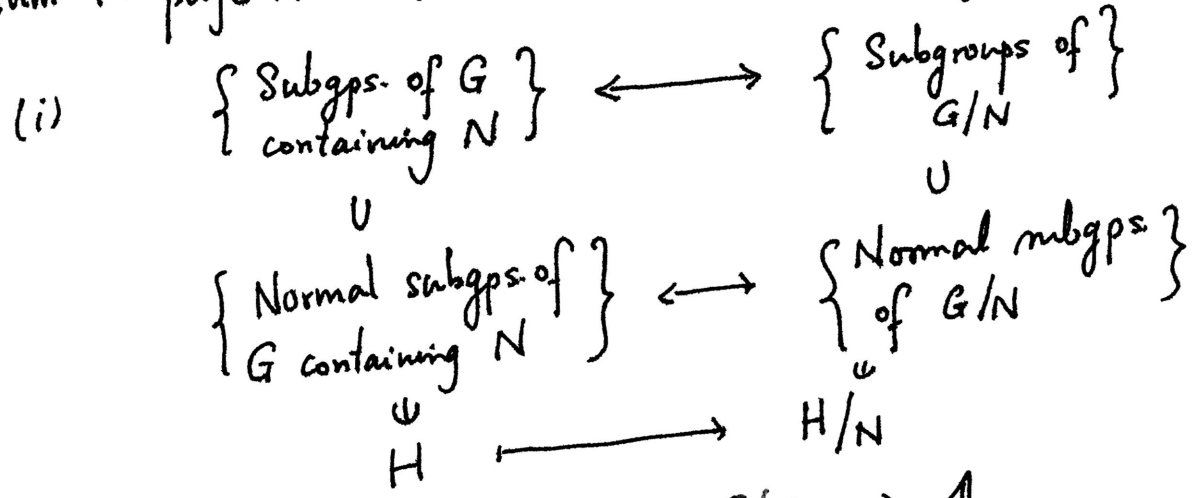
# Lecture 5

(5.0) Recall: last time we proved some isomorphism theorems

0. (Thm 3.3 page 3).  $\varphi: G \rightarrow G'$  gp. hom. Then

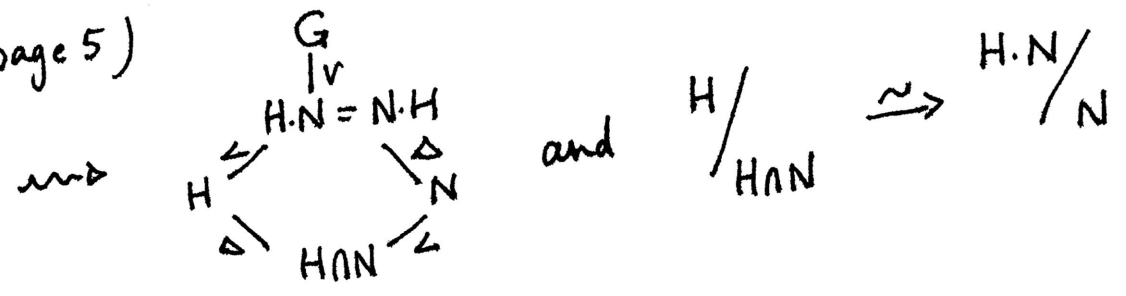


1. (Thm 4.1 page 1)  $N \triangleleft G$  (normal subgroup).



2. (Thm 4.3 page 5)

$H < G$   
 $N \triangleleft G$



•  $G = H \rtimes N$  means (i)  $H < G$  (subgroup),  $N \triangleleft G$  (normal)

(ii)  $H \cdot N = G$  and (iii)  $H \cap N = \{e\}$

Equivalently, there is a split short exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \mathbb{1}$$

$\longleftarrow \overset{\varepsilon}{\text{---}} \longleftarrow$  (dashed arrow from  $H$  to  $G$ )

(5.1) Definition. Let  $\mathbb{1} \rightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \rightarrow \mathbb{1}$  be a short exact sequence.

A retraction (or projection) is a group hom.  $r: G \rightarrow N$  s.t.  $r \circ \varphi = \text{Id}_N$  (in particular  $r$  is surjective).

Prop. If a short exact sequence  $\mathbb{1} \rightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \rightarrow \mathbb{1}$  admits a retraction  $r: G \rightarrow N$ , then  $G \cong H \times N$  (direct product)

$$\begin{array}{ccccccc}
 \mathbb{1} & \rightarrow & N & \xrightarrow{i} & H \times N & \xrightarrow{\pi} & H \rightarrow \mathbb{1} \\
 & & \parallel & & \uparrow \eta \cong & & \parallel \\
 \mathbb{1} & \rightarrow & N & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H \rightarrow \mathbb{1} \\
 & & & & \downarrow \eta \cong & & \\
 & & & & \mathbb{1} & & 
 \end{array}$$

$\longleftarrow \overset{p}{\text{---}} \longleftarrow$  (dashed arrow from  $H \times N$  to  $N$ )  
 $\longleftarrow \overset{r}{\text{---}} \longleftarrow$  (dashed arrow from  $G$  to  $N$ )

•  $i: N \rightarrow H \times N$  is the natural inclusion

•  $\pi: H \times N \rightarrow H$  and  $p: H \times N \rightarrow N$  are natural surjections.

•  $\eta : G \rightarrow H \times N$  is given by

$$\eta(g) = (\psi(g), r(g))$$

Proof. •  $\eta$  is a group hom. since both  $\psi$  and  $r$  are.

•  $\eta$  is surjective: let  $h \in H$  and  $x \in N$ .

Choose  $g \in G$  s.t.  $\psi(g) = h$  (exists since  $\psi$  is surjective)

Take  $\tilde{g} := g \cdot \varphi(r(g))^{-1} \cdot \varphi(x) \in G$  Then

$$\psi(\tilde{g}) = \psi(g) = h$$

(as  $\psi(\varphi(z)) = e$   
 $\forall z \in N$ )

$$\text{and } r(\tilde{g}) = r(g) r(g)^{-1} x = x$$

(because  $r(\varphi(z)) = z$   
 $\forall z \in N$ )

•  $\text{Ker}(\eta) = \{e\}$ . If  $\eta(g) = (e, e)$ , then

$$\psi(g) = e \text{ and } r(g) = e. \text{ But } \psi(g) = e \Leftrightarrow g \in \text{Ker}(\psi) = \text{Im}(\varphi)$$

i.e.  $g = \varphi(x)$  for some  $x \in N$ .

$$\Rightarrow r(g) = r(\varphi(x)) = x = e \Rightarrow g = \varphi(e) = e$$

Thus (by ~~Thm 3.3 (page 3)~~)  $\eta$  is an iso.

Commutativity of the diagram is a straight-forward check  $\square$

(5.2) Summary.

(4)

Split short exact sequences  $\longleftrightarrow$  Semi direct product  
(ie. a section exists)

Trivial short exact sequence  $\longleftrightarrow$  Direct Product  
(retraction exists)

(5.3) Constructing semidirect products

Input:  $H, N$  two groups.  $\alpha: H \longrightarrow \text{Aut}_{\text{Gps}}(N)$   
a group hom.

Define a group, say  $G$  as follows

$G = H \times N$  as a set.

$\forall h_1, h_2 \in H, x_1, x_2 \in N$

Law of composition  $(h_1, x_1) \cdot (h_2, x_2) = (h_1 h_2, \alpha(h_2^{-1})(x_1) x_2)$

[Check: law of composition is associative:

$$((h_1, x_1) \cdot (h_2, x_2)) \cdot (h_3, x_3) = (h_1 h_2 h_3, \alpha(h_3^{-1})(\alpha(h_2^{-1})(x_1) \cdot x_2) \cdot x_3)$$

$$(h_1, x_1) \cdot ((h_2, x_2) \cdot (h_3, x_3)) = (h_1 h_2 h_3, \alpha((h_2 h_3)^{-1})(x_1) \cdot \alpha(h_3^{-1})(x_2) \cdot x_3)$$

are equal since  $\alpha$  is a group hom &  $\forall h \in H, \alpha(h): N \rightarrow N$  is a gp. hom.]

$(e_H, e_N) \in G$  is the identity element

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$$(h, x)^{-1} = (h^{-1}, \alpha(h)(x^{-1})) \quad : \quad \underline{\text{check}} :$$

$$\begin{aligned} (h, x) \cdot (h^{-1}, \alpha(h)(x^{-1})) &= (h \cdot h^{-1}, \alpha(h)(x) \cdot \alpha(h)(x^{-1})) \\ &= (e_H, \alpha(h)(x \cdot x^{-1})) = (e_H, e_N) \quad \square \end{aligned}$$

(5.4)

Lemma. The group  $G$  defined above is a semidirect product of  $H$  and  $N$ , denoted by  $H \rtimes_{\alpha} N$ .

Conversely, if  $G$  is a semidirect product of  $H$  &  $N$ ,

then  $\exists$  a group hom.  $\alpha: H \rightarrow \text{Aut}(N)$  s.t.

Groups

$$G \cong H \rtimes_{\alpha} N.$$

Proof. Let  $G$  be the group defined in §5.3 above.

(i)  $H \hookrightarrow G$  is a subgroup (clear from defn.)  
 $h \mapsto (h, e_N)$

$N \hookrightarrow G$  is a normal subgroup:  
 $x \mapsto (e_H, x)$

$$\begin{aligned} (h, e) (e, x) (h^{-1}, e) &= (h, x) (h^{-1}, e) \\ &= (e, \alpha(h)(x)) \in N \quad \square \end{aligned}$$

(ii)  $G = H \cdot N$  because  $(h, x) = (h, e) \cdot (e, x)$ . (6)

(iii)  $H \cap N = \{e\}$  clear from definition.

Conversely, assume  $G$  is a semidirect product of  $H$  &  $N$ .

Define  $\alpha: H \longrightarrow \text{Aut}(N)$  as:  $\forall h \in H, x \in N$ :

$$\alpha(h)(x) := hxh^{-1} \in N.$$

[  $\forall h \in H$ ,  $\alpha(h)$  is a group iso. and  $\alpha$  is a group hom  $\leftarrow$  see problem #12 of Set 1 ]

Define  $\eta: H \rtimes_{\alpha} N \longrightarrow G$  by  
 $\eta((h, x)) = hx \quad (\forall h \in H, x \in N).$

•  $\eta$  is a group hom:

$$\eta((h_1, x_1) \cdot (h_2, x_2)) = \eta((h_1 h_2, \alpha(h_2^{-1})(x_1 \cdot x_2)))$$

$$= h_1 h_2 \cdot h_2^{-1} x_1 h_2 x_2 = h_1 x_1 h_2 x_2$$

$$= \eta((h_1, x_1)) \cdot \eta((h_2, x_2))$$

•  $\eta$  is surjective: because  $G = H \cdot N$ .

•  $\text{Ker}(\eta) = \{(e, e)\}$  :

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if  $\eta((h, x)) = e_G$  then  $hx = e_G$ , i.e.

$x = h^{-1} \in N \Rightarrow$  both  $h$  and  $x$  are in  $H \cap N = \{e\}$

$\Rightarrow h = e_H, x = e_N.$  □

(5.5) In conclusion

Semidirect products  
of  $H$  and  $N$

$\longleftrightarrow$

Group homomorphisms  
 $H \rightarrow \text{Aut}_{\text{Group}}(N)$

$\Downarrow$

Direct  $\Downarrow$  product

Trivial hom

$H \rightarrow \text{Aut}(N)$

$\Downarrow$   
 $h \mapsto \text{Id}_N (\forall h \in H)$