Lecture 5

(5.0) **Recall**: last time we proved some isomorphism theorems

0. (Thm 3.3 page 3). $\varphi : G \longrightarrow G'$ gp. hom. Then

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, two heads] & G' \\ G/_{\mathrm{Ker}(\varphi)} \arrow[r, "\sim", "\overline{\varphi}"'] & \mathrm{Im}(\varphi) \arrow[u, hook] \end{tikzcd}$$

1. (Thm 4.1 page 1) $N \triangleleft G$ (normal subgroup).

(i)
$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Subgps. of } G \\ \text{containing } N \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{Subgroups of} \\ G/N \end{array} \right\} \\ \cup & & \cup \\ \left\{ \begin{array}{c} \text{Normal subgps. of} \\ G \text{ containing } N \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{Normal subgps.} \\ \text{of } G/N \end{array} \right\} \\ \Psi & & \Psi \\ H & \longmapsto & H/N \end{array}$$

(ii)
$$\begin{tikzcd} \mathbb{1} \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/H \arrow[r] \arrow[d, "\cong", "\text{iso}"'] & \mathbb{1} \\ \mathbb{1} \arrow[r] & H/N \arrow[r] & G/N \arrow[r] & (G/N)/_{(H/N)} \arrow[r] & \mathbb{1} \end{tikzcd}$$

2. (Thm 4.3 page 5)

$H < G$
$N \triangleleft G$ $\rightsquigarrow$

$$\begin{tikzcd} & G \arrow[d, phantom, "\geq"] & \\ & H\cdot N = N\cdot H & \\ H \arrow[ur, phantom, "\leq"] & & N \arrow[ul, phantom, "\triangleleft"] \\ & H \cap N \arrow[ul, phantom, "\triangleleft"] \arrow[ur, phantom, "\leq"] & \end{tikzcd}$$

and $H/_{H\cap N} \xrightarrow{\sim} H\cdot N/_N$

- $G = H \ltimes N$ means (i) $H < G$ (subgroup), $N \lhd G$ (normal)

  (ii) $H \cdot N = G$ and (iii) $H \cap N = \{e\}$

Equivalently, there is a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\varphi"'] & G \arrow[r, "\psi"'] & H \arrow[l, bend right, dashed, "s"'] \arrow[r] & 1 \end{tikzcd}$$

(5.1) Definition. Let $1 \to N \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$ be a short exact sequence. A retraction (or projection) is a group hom. $r : G \to N$ s.t. $r \circ \varphi = \mathrm{Id}_N$ (in particular r is surjective).

Prop. If a short exact sequence $1 \to N \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$ admits a retraction $r : G \to N$, then $G \simeq H \times N$ (direct product)

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & H \times N \arrow[r, "\pi"] \arrow[l, bend right, dashed, "\rho"'] \arrow[d, leftrightarrow, "\eta"', "\cong"] & H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] \arrow[l, bend left, dashed, "r"] & H \arrow[r] & 1 \end{tikzcd}$$

.

- $i : N \to H \times N$ is the natural inclusion

- $\begin{tikzcd} & H \times N \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ H & & N \end{tikzcd}$ are natural surjections.

• $\eta : G \longrightarrow H \times N$ is given by

$$\eta(g) = (\psi(g), r(g))$$

Proof. • η is a group hom. since both ψ and r are.

• η is surjective: let $h \in H$ and $x \in N$.

Choose $g \in G$ s.t. $\psi(g) = h$ (exists since ψ is surjective)

Take $\tilde{g} := g \cdot \varphi(\overset{r}{\cancel{\psi}}(g))^{-1} \cdot \varphi(x) \in G$ Then

$\psi(\tilde{g}) = \psi(g) = h$ (as $\psi(\varphi(z)) = e$ $\forall z \in N$) and $r(\tilde{g}) = r(g)\, r(g)^{-1} x = x$ (because $r(\varphi(z)) = z$ $\forall z \in N$)

• $Ker(\eta) = \{e\}$. If $\eta(g) = (e, e)$, then

$\psi(g) = e$ and $r(g) = e$. But $\psi(g) = e \iff g \in Ker(\psi)$ $= Im(\varphi)$

i.e. $g = \varphi(x)$ for some $x \in N$.

$\Rightarrow r(g) = r(\varphi(x)) = x = e \Rightarrow g = \varphi(e) = e$

Thus (~~by Thm 3.3 (page 3)~~) η is an iso.

· Commutativity of the diagram is a straightforward check

□

(5.2) Summary.

Split short exact sequences (ie. a section exists) $\longleftrightarrow$ Semi direct product

Trivial short exact sequence (retraction exists) $\longleftrightarrow$ Direct Product

(5.3) Constructing semidirect products

Input : H, N two groups. $\alpha : H \longrightarrow \mathrm{Aut}_{\mathrm{Gps}}(N)$ a group hom.

Define a group, say G as follows

$G = H \times N$ as a set. $\forall\, h_1, h_2 \in H,\ x_1, x_2 \in N$

Law of composition $\quad (h_1, x_1) \cdot (h_2, x_2) = (h_1 h_2\, ,\ \alpha(h_2^{-1})(x_1)\, x_2)$

[Check : law of composition is associative :

$$((h_1, x_1)\cdot(h_2, x_2))\cdot(h_3, x_3) = \left(h_1 h_2 h_3\, ,\ \alpha(h_3^{-1})\left(\alpha(h_2^{-1})(x_1)\cdot x_2\right)\cdot x_3\right)$$

$$(h_1, x_1)\cdot((h_2, x_2)\cdot(h_3, x_3)) = \left(h_1 h_2 h_3\, ,\ \alpha((h_2 h_3)^{-1})(x_1)\cdot \alpha(h_3^{-1})(x_2)\cdot x_3\right)$$

are equal since α is a group hom & $\forall\, h \in H$, $\alpha(h) : N \to N$ is a gp. hom.]

$(e_H, e_N) \in G$ is the identity element

$(h, x)^{-1} = (h^{-1}, \alpha(h)(x^{-1}))$: check :

$$(h, x) \cdot (h^{-1}, \alpha(h)(x^{-1})) = (h \cdot h^{-1}, \alpha(h)(x) \cdot \alpha(h)(x^{-1}))$$

$$= (e_H, \alpha(h)(x \cdot x^{-1})) = (e_H, e_N) \quad \square$$

(5.4)

Lemma. The group G defined above is a semidirect product of H and N, denoted by $H \ltimes_\alpha N$.

Conversely, if G is a semidirect product of H & N, then $\exists$ a group hom. $\alpha : H \longrightarrow \mathrm{Aut}_{\text{Groups}}(N)$ s.t.

$G \simeq H \ltimes_\alpha N$.

Proof. Let G be the group defined in §5.3 above.

(i) $\begin{array}{l} H \hookrightarrow G \\ h \mapsto (h, e_N) \end{array}$ is a subgroup (clear from defn.)

$\begin{array}{l} N \hookrightarrow G \\ x \longmapsto (e_H, x) \end{array}$ is a normal subgroup :

$$(h, e)(e, x)(h^{-1}, e) = (h, x)(h^{-1}, e)$$

$$= (e, \alpha(h)(x)) \in N \quad \square$$

(ii) $G = H \cdot N$ because $(h, x) = (h, e) \cdot (e, x)$.

(iii) $H \cap N = \{e\}$ clear from definition.

Conversely, assume G is a semidirect product of H & N.

Define $\alpha : H \longrightarrow \mathrm{Aut}(N)$ as: $\forall h \in H, x \in N$:

$$\alpha(h)(x) := h x h^{-1} \in N.$$

[$\forall h \in H$, $\alpha(h)$ is a group iso. and α is a group hom $\leftarrow$ see problem #12 of Set 1]

Define $\eta : H \rtimes_{\alpha} N \longrightarrow G$ by

$$\eta((h, x)) = h \cdot x \quad (\forall h \in H, x \in N).$$

- η is a group hom:

$$\eta\big((h_1, x_1) \cdot (h_2, x_2)\big) = \eta\big((h_1 h_2, \alpha(h_2^{-1})(x_1) \cdot x_2)\big)$$

$$= h_1 h_2 \cdot h_2^{-1} x_1 h_2 x_2 = h_1 x_1 h_2 x_2$$

$$= \eta((h_1, x_1)) \cdot \eta((h_2, x_2))$$

- η is surjective: because $G = H \cdot N$.

- $\mathrm{Ker}(\eta) = \{(e,e)\}$:

if $\eta((h,x)) = e_G$ then $hx = e_G$, i.e.

$x = h^{-1} \in N \implies$ both h and x are in $H \cap N = \{e\}$

$\implies h = e_H$, $x = e_N$. $\square$

(5.5) In conclusion

Semidirect products of H and N $\longleftrightarrow$ Group homomorphisms $H \longrightarrow \mathrm{Aut}_{\text{Group}}(N)$

$\Downarrow$ Direct product

$\Downarrow$ Trivial hom

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Aut}(N) \\ \Downarrow & & \Downarrow \\ h & \longmapsto & \mathrm{Id}_N \ (\forall h \in H) \end{array}$$