(8.0) Definitions. A group S is said to be <u>simple</u> if $\{e\}$ and S are the only normal subgroups of S.

e.g. A_n $(n \geq 5)$ are simple (next week)

$\mathbb{Z}/p\mathbb{Z}$ (where p is a prime) are simple.

$PSL_n = SL_n / Z(SL_n)$ are simple.

(8.1) Let G be a group. A <u>composition series</u> of G is a finite sequence of subgroups of G

$$\Sigma : \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that:

$G_{j+1} \lhd G_j$ is normal $\forall\ 0 \leq j \leq k-1$

The successive quotients: $gr_i(G) := G_i / G_{i+1}$ $(0 \leq i \leq k-1)$

(or $gr_i^{\Sigma}(G)$ if the composition series Σ is not clear from the context)

A composition series Σ' is said to be <u>finer</u> than Σ if Σ is obtained from Σ' by omitting some terms.

More precisely : $\Sigma' : \; G = G'_0 \supset \dots \supset G'_m = \{e\}$

$\Sigma : \; G = G_0 \supset \dots \supset G_n = \{e\}$

Σ' is finer than Σ if $n \leq m$ and $\exists$ an order-preserving injective map $\phi : \{0, \dots, n\} \longrightarrow \{0, \dots, m\}$ such that

$$G_j = G'_{\phi(j)} \quad \forall \; 0 \leq j \leq n$$

Remark : In general a series obtained from a composition series Σ' by omitting some terms is not a composition series, since for $j > i+1$, G'_j is not in general a normal subgroup of G'_i.

Let $\Sigma_1 : \; G = G_0 \supset \dots \supset G_m = \{e\}$

$\Sigma_2 : \; H = H_0 \supset \dots \supset H_n = \{e\}$

be two composition series. We say Σ_1 and Σ_2 are <u>equivalent</u> if $m = n$ and $\exists$ $\sigma \in S_n$ s.t. permutations of $[0, n-1]$

$$gr_i^{\Sigma_1}(G) = gr_{\sigma(i)}^{\Sigma_2}(H) \quad \forall \; i$$

e.g. $\mathbb{Z}/4\mathbb{Z} \supset \mathbb{Z}/2\mathbb{Z} \supset \{e\}$
$(\mathbb{Z}/2\mathbb{Z})^2 \supset \mathbb{Z}/2\mathbb{Z} \supset \{e\}$ are equivalent

(8.2) Theorem (Schrier) Let Σ_1 and Σ_2 be two composition series of a group G. Then there exist composition series Σ_1' and Σ_2' finer than Σ_1 and Σ_2 respectively, such that Σ_1' and Σ_2' are equivalent.

[read as: any two composition series have a common refinement]

Proof. Let $\Sigma_1 : G = H_0 \supset \dots \supset H_n = \{e\}$
$\Sigma_2 : G = K_0 \supset \dots \supset K_p = \{e\}$

Idea: use Σ_2 to insert $p-1$ subgroups, say $H'_{i,j}$ between H_i and H_{i+1} $(\forall i, 0 \leq i \leq n-1)$ $(1 \leq j \leq p-1)$ $\rightsquigarrow$ Σ_1' finer than Σ_1.

Similarly use Σ_1 to insert $n-1$ subgroups, say $K'_{j,i}$ $(1 \leq i \leq n-1)$

between K_j and K_{j+1} $(\forall j : 0 \leq j \leq p-1)$ $\leadsto \Sigma_2'$ finer than Σ_2.

Finally show that $\Sigma_1' \sim \Sigma_2'$.

So, for each i and j define
$$H_{i,j}' = H_{i+1} \cdot (H_i \cap K_j)$$
$$K_{j,i}' = K_{j+1} \cdot (H_i \cap K_j)$$

It is clear that

- $H_{i,0}' = H_i$; $H_{i,p}' = H_{i+1}$; $K_{j,0}' = K_j$; $K_{j,n}' = K_{j+1}$
- $H_{i,j+1}' < H_{i,j}'$; $K_{j,i+1}' < K_{j,i}'$ (subgroups)

Remains to show

(TS) (i) $H_{i,j+1}' \lhd H_{i,j}'$ and $K_{j,i+1}' \lhd K_{j,i}'$ are normal subgroups.

(ii) $$H_{i,j}' / H_{i,j+1}' \simeq K_{j,i}' / K_{j,i+1}'$$

For simplicity of notation write
$$H = H_i \rhd H' = H_{i+1}$$
$$K = K_j \rhd K' = K_{j+1}$$

Then the theorem is proved using the following lemma. $\square$

(8.3) Lemma (Zassenhaus)

Let G be a group; H, K be two subgroups of G,

$H' \lhd H$ two normal subgroups of H & K respectively.
$K' \lhd K$

Then (i) $H'.(H \cap K') \lhd H'.(H \cap K)$
$K'.(K \cap H') \lhd K'.(K \cap H)$ are normal subgroups.

(ii) $$H'.(H \cap K) \big/ H'.(H \cap K') \simeq K'.(K \cap H) \big/ K'.(K \cap H')$$

Proof. The picture on the next page outlines the main steps of the proof.

Step 1. $(H' \cap K).(K' \cap H)$ is a normal subgroup of $H \cap K$.

This is true since both $H' \cap K$ and $K' \cap H$ are normal subgroups of $H \cap K$.

$$
\begin{tikzcd}[arrows=dash]
 & H \arrow[d] & & & & K \arrow[d] & \\
 & H'\cdot(H\cap K) \arrow[dd, "\triangleright"] \arrow[drr] & & & & K'\cdot(K\cap H) \arrow[dd, "\triangleright"] \arrow[dll] & \\
 & & & H\cap K \arrow[dd, "\triangleright"] & & & \\
 & H'\cdot(H\cap K') \arrow[drr] \arrow[dl] & & & & K'\cdot(K\cap H') \arrow[dll] \arrow[dr] & \\
H' \arrow[dr] & & & (H'\cap K)\cdot(K'\cap H) \arrow[dll] \arrow[drr] & & & K' \arrow[dl] \\
 & H'\cap K & & & & K'\cap H &
\end{tikzcd}
$$

Step 2. $H' \cdot (H \cap K')$ is a normal subgroup of $H' \cdot (H \cap K)$.

Consider the following more general claim:

[Claim 1. If $\mathcal{G}$ is a group, $\underset{\text{subgp.}}{G_1 < \mathcal{G}}$, $\underset{\text{normal}}{N \triangleleft \mathcal{G}}$ and $\underset{\text{normal}}{G_2 \triangleleft G_1}$; then $G_2 \cdot N$ is normal subgp. of $G_1 \cdot N$.

Now the assertion of Step 2 follows by taking $\mathcal{G} = H$, $N = H'$, $G_1 = H \cap K$ and $G_2 = H \cap K'$.

Step 3. Now we use the second iso. thm (Theorem 4.3 p. 5)

$$H' \cdot (H \cap K) \big/ H' \cdot (H \cap K') \simeq H \cap K \big/ (H \cap K) \cap (H' \cdot (H \cap K'))$$

[Claim 2. $(H \cap K) \cap (H' \cdot (H \cap K')) = (H' \cap K) \cdot (K' \cap H)$

Assuming this claim, we obtain an isomorphism

$$H' \cdot (H \cap K) \big/ H' \cdot (H \cap K') \simeq H \cap K \big/ (H' \cap K) \cdot (K' \cap H)$$

Carrying out these steps with the roles of H & K reversed we also get $K' \cdot (K \cap H) \big/ K' \cdot (K \cap H') \simeq K \cap H \big/ (K' \cap H) \cdot (H' \cap K)$

and the lemma follows.

Proof of Claim 1. Let $\pi : \mathcal{G} \to \mathcal{G}/N$ be the natural projection and let $\overline{G}_1 = \pi(G_1)$. Restricting to G_1, we get a surjective map $\pi : G_1 \to \overline{G}_1$ hence $\overline{G}_2 := \pi(G_1)$ is a normal subgroup of $\overline{G}_1$ (since π is surjective).

Now consider the homomorphism defined by

$$G_1 \cdot N \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/N$$

$$\alpha : G_1 \cdot N \longrightarrow \overline{G}_1$$

Then $\alpha^{-1}(\overline{G}_2) = G_2 \cdot N$ is inverse image of a normal subgroup, hence normal.

Proof of Claim 2. $(H' \cap K) \cdot (K' \cap H) \subset (H \cap K) \cap (H' \cdot (H \cap K'))$ is obvious.

Conversely, let $x = a \cdot b \in (H' \cdot (H \cap K')) \cap (H \cap K)$ where $a \in H'$ and $b \in H \cap K'$ $(\subset H \cap K)$

So $a = x \cdot b^{-1} \in H \cap K$

$\Rightarrow a \in H' \cap (H \cap K) = H' \cap K$

Hence $x \in (H' \cap K) \cdot (H \cap K')$ $\square$