Lecture 9

(9.0) Recall that last time we defined a composition series of a group G to be a sequence of subgroups

$$\Sigma: \quad G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$$

such that G_{j+1} is a normal subgroup of G_j ($\forall\, 0 \le j \le n-1$).

We proved (Theorem (8.2) p. 3) that any two composition series Σ_1 and Σ_2 of a group G admit refinements Σ_1' and Σ_2' resp. which are equivalent.

(9.1) Definition. A composition series $\Sigma: G = G_0 \supset \dots \supset G_n = \{e\}$ is said to be a <u>Jordan-Hölder</u> series if

(i) Σ is strictly decreasing (i.e. $G_{j+1} \subsetneqq G_j$ $\forall\, 0 \le j \le n-1$)

(ii) There is no strictly decreasing composition series distinct from Σ and finer than Σ.

Proposition. A composition series Σ of G is Jordan-Hölder (or JH for short) if and only if $gr_i^{\Sigma}(G)$ is simple $\forall\, 0 \le i \le n-1$

(recall: $\{e\}$ is <u>not</u> simple).

Proof. Note that a composition series is strictly decreasing if and only if none of its associated quotients is $\{e\}$.

Let $\Sigma : G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{e\}$ be a strictly decreasing composition series that is not JH. Then $\exists$ a strictly decreasing series Σ' finer than Σ. Thus $\exists\ 0 \leq i \leq n-1$ such that $G_{i+1} \underset{\neq}{\trianglelefteq} G_i$ are not consecutive in Σ'. That is, there exists an intermediate normal subgroup $G_{i+1} \underset{\neq}{\trianglelefteq} H \underset{\neq}{\trianglelefteq} G_i$. Hence H/G_{i+1} is a non-trivial proper normal subgroup of G_i/G_{i+1} which is thus not simple.

Conversely assume $\Sigma : G = G_0 \supsetneq \cdots \supsetneq G_n = \{e\}$ is a strictly decreasing composition series, one of whose graded pieces, say G_i/G_{i+1}, is not simple. By the first isomorphism theorem, a proper, nontrivial normal subgroup of G_i/G_{i+1} is of the form H/G_{i+1} for some intermediate normal subgroup $G_{i+1} \underset{\neq}{\trianglelefteq} H \underset{\neq}{\trianglelefteq} G_i$. Thus we get a distinct, strictly decreasing series $G = G_0 \supset \cdots \supset G_i \supset H \supset G_{i+1} \supset \cdots \supset G_n = \{e\}$ finer than Σ. Hence Σ is not J.H.

$\square$

(9.2) A general group G need not possess any Jordan-Hölder series. e.g. $\mathbb{Z} \supsetneq 2\cdot\mathbb{Z} \supsetneq 4\cdot\mathbb{Z} \supsetneq 8\cdot\mathbb{Z} \supsetneq \cdots$ cannot terminate.

However every finite group G has a Jordan-Hölder series.
To see this, let H_1 be maximal among (proper) normal subgroups of G.
Recursively let H_{n+1} be maximal among proper normal subgroups of H_n.
This procedure must halt, at most $|G|$ steps later, thus forming a Jordan-Hölder series.

(9.3) Theorem (Jordan-Hölder) Two Jordan-Hölder series of a group are equivalent.

Proof. Let Σ_1 and Σ_2 be two Jordan-Hölder series of a group G. Using Theorem (8.2) (p.3), we obtain two composition series Σ_1' and Σ_2' finer than Σ_1 & Σ_2 respectively, which are equivalent.
As Σ_1 (and Σ_2) is J.H. Σ_1' (and Σ_2') is either identical with Σ_1 (resp. Σ_2) or obtained from Σ_1 (resp. Σ_2) by ~~inserting~~ repeating some terms. As the series of quotients of Σ_1' & Σ_2' differ only in the order of terms, the same is true for Σ_1 & Σ_2.

□

(9.4) Corollary. Let G be a group that admits a J-H series. If Σ is any strictly decreasing composition series of G, then there exists a J-H series finer than Σ.

Sketch of a proof. Let Σ_0 be a J-H series of G. By Theorem 8.2 (p.3), we can find Σ_0', Σ' two equivalent composition series refining Σ_0 and Σ respectively. The same argument as in the proof of Thm 9.3 above allows us to obtain a J-H series from Σ' finer than Σ. □

(9.5) Derived series of a group.

Lemma. Let G be a group and $A \lhd G$, $B \lhd G$ two normal subgroups. Let (A,B) be the subgroup of G generated by the subset $\{aba^{-1}b^{-1} : a \in A, b \in B\}$.
Then (A,B) is normal in G.

Proof $\forall\, g \in G$, $a \in A$, $b \in B$

$$g\,(aba^{-1}b^{-1})\,g^{-1} = \underbrace{(gag^{-1})}_{\substack{\in A \\ \text{(as } A \text{ is normal)}}}\underbrace{(gbg^{-1})}_{\in B \text{ as } B \text{ is normal}}(gag^{-1})^{-1}(gbg^{-1})^{-1}$$

□

Cor. Define recursively $D^0(G) = G$ and

$$D^{n+1}(G) \; (= D(D^n(G))) := (D^n(G), D^n(G)).$$

Then each $D^n(G)$ is normal in G;

$$G = D^0(G) \supset D^1(G) \supset \dots \supset D^n(G) \supset D^{n+1}(G) \supset \dots$$

and $D^n(G) / D^{n+1}(G)$ is abelian $\forall\, n \geq 0$.

[The last assertion follows from Problem 11 of Set 1]

Definition: $G = D^0(G) \supset D^1(G) \supset \dots$ is called the <u>derived series</u> of G.

(9.6) Definition: We say G is <u>solvable</u> if there exists $N \geq 0$ such that $D^N(G) = \{e\}$.

Remarks. • The term "solvable" originates from Galois theory (next course)

- $D^0(G) = \{e\} \iff G$ is trivial.
- $D^1(G) = \{e\} \iff G$ is abelian (Hence all abelian groups are solvable.)

- The group of upper triangular matrices is solvable

e.g.
$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \begin{matrix} a, d \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \\ b \in \mathbb{C} \end{matrix} \right\}$$
$$\cup$$
$$D(B) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C} \right\}$$
$$\cup$$
$$D^2(B) = \{e\}$$

- If G is non-abelian and simple, then

$D(G) \neq \{e\}$ (o/w G would be abelian)

$D(G) \underset{\text{normal}}{\lhd} G \Rightarrow D(G) = G$

so $D^n(G) = G \;\; \forall n \geq 0$ hence G is not solvable.