Lecture 10

(10.0) Recall that last time we defined the following:

- for a group G and a pair A, B of its <u>normal</u> subgroups,

  (A, B) = subgroup of G generated by the set
  $$\{aba^{-1}b^{-1} : a \in A, b \in B\}$$
  (we saw that (A, B) is normal.)

- $D(G) := (G, G)$.

Main property of $D(G)$: (Problem 11 of Set 1; Problem 1 of Set 2)
$G/D(G)$ is abelian. Hence any subgroup of G containing $D(G)$ is normal. Conversely, if $H \lhd G$ s.t. G/H is abelian then $D(G) \subset H$ (use $G_1 = G$, $G_2 = G/H$ in Problem 1 of Set 2).

- G is said to be <u>solvable</u> if $\exists\, N \geq 0$ such that
$$D^N(G) = \{e\}.$$

Recall that $\{D^n(G)\}_{n \geq 0}$ were defined by:
$$D^0(G) = G \qquad D^{n+1}(G) = D(D^n(G)) = (D^n(G), D^n(G))$$

Thus a solvable group G has a composition series

$$G = D^0(G) \supset D^1(G) \supset \dots \supset D^N(G) = \{e\} \quad \text{with}$$

- each $D^j(G)$ is normal subgroup of G.
- associated graded $D^j(G)/D^{j+1}(G)$ is abelian $\forall\, 0 \leq j \leq N-1$.

(10.1) Theorem. Let G be a group and assume it has a composition series

$$\Sigma : \; G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$$

such that $gr_j^{\Sigma}(G) = G_j / G_{j+1}$ is abelian. Then G is solvable.

Proof. We will prove by induction on j that $D^j(G) \subset G_j$.
Thus establishing that $D^n(G) \subset G_n = \{e\}$, hence G is solvable.

Base case : $j=0$ is obvious $(D^0(G) = G = G_0)$.

Induction step. Assume $D^j(G) \subset G_j$. Since G_j/G_{j+1} is abelian, $D(G_j) \subset G_{j+1}$ (see previous page).

$$\Longrightarrow D^{j+1}(G) = D(D^j(G)) \subset D(G_j) \subset G_{j+1}$$

and we are done. $\square$

(10.2) Corollary. (i) G is solvable $\Longleftrightarrow$ it has a composition series whose associated graded pieces are abelian.

(ii) Every p-group is solvable (see Theorem 7.5 p.5).

(10.3) Another application of Thm. 10.1 (p.2) is the following

Proposition. Let G be a group and $N \triangleleft G$ a normal subgroup. Then G is solvable if and only if N and G/N are solvable.

[This statement is usually phrased as follows:
If $\mathbb{1} \to G_1 \to G_2 \to G_3 \to \mathbb{1}$ is an exact sequence, then G_2 is solvable $\Longleftrightarrow$ G_1 & G_3 are solvable]

Proof. ($\Rightarrow$) Let us assume G is solvable and $n \geq 0$ is such that $D^n(G) = \{e\}$.

Then $\underbrace{D^n(N) \subset D^n(G)}_{\text{clear by an easy induction argument on } n.} = \{e\} \Rightarrow N$ is solvable.

If $\pi : G \to G/N$ is the natural projection, then

- $\pi(D(G)) = \pi((G,G)) = (\pi(G), \pi(G)) = D(G/N)$

repeatedly using this we get

$$D^n(G/N) = \pi(D^n(G)) = \{\pi(e)\} = \{e_{G/N}\}$$

$\Rightarrow$ G/N is solvable.

($\Leftarrow$) Now assume N & G/N are solvable. By Theorem 10.1 and Cor. 10.2; we have composition series with abelian graded pieces

$$\Sigma : \quad N = N_0 \supset N_1 \supset \dots \supset N_k = \{e\}$$

$$\overline{\Sigma} : \quad G/N = \overline{G}_0 \supset \overline{G}_1 \supset \dots \supset \overline{G}_\ell = \{e\}$$

Set $G_j = \pi^{-1}(\overline{G}_j)$ $(0 \leq j \leq \ell)$. Note that $G_\ell = N$.

Define $G_{\ell+i} := N_i$ $(1 \leq i \leq k)$.

<u>Claim</u>: $\forall\ 0 \leq j \leq \ell - 1$, $G_j / G_{j+1} \xrightarrow[\text{iso.}]{\cong} \overline{G}_j / \overline{G}_{j+1}$

Pf. of the claim. $\pi\big|_{G_j} : G_j \longrightarrow \overline{G}_j$ is surjective by definition. Let $\alpha : G_j \to \overline{G}_j / \overline{G}_{j+1}$ be its composition with the natural surjection. Then $\mathrm{Ker}(\alpha) = G_{j+1}$ and the claim follows from the fundamental theorem of group hom-s.

Hence $G = G_0 \supset G_1 \supset \dots \supset G_{k+\ell} = \{e\}$ is a composition series with abelian graded pieces $\underset{\text{(Thm. 10.1 above)}}{\Longrightarrow}$ G is solvable $\square$

(10.4) Assume G is finite and solvable. Then G admits a J-H series (being finite), say Σ_1, and a composition series with abelian factors, say Σ_2. Using Schrier's Thm (8.2 on p. 3), we can find their respective refinements Σ_1' and Σ_2' which are equivalent.

So, the graded pieces of Σ_2' are abelian (being a refinement of Σ_2), and the same must be true for Σ_1' (being equivalent to Σ_1') and also for Σ_1 (being J-H whose refinement has abelian graded pieces).

$\Rightarrow$ $gr_j^{\Sigma_1}(G)$ is simple (since Σ_1 is J-H) and abelian

$\Rightarrow$ $gr_j^{\Sigma_1}(G) \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p. Thus,

Prop. If G is finite then G is solvable if and only if $gr_j^{\Sigma}(G)$ is cyclic of prime order ($\forall\, 0 \le j \le$ length of Σ) for some (and therefore any) J-H series Σ of G.

(10.5) Lower central series.

Define recursively $C^{n+1}(G) = (G, C^n(G))$ (again a normal subgroup of G)

$$C^1(G) = G$$

Definition. G is said to be nilpotent if $\exists\ n \geq 1$ such that $C^n(G) = \{e\}$

Thus, a nilpotent group G admits a composition series

$$G = C^1(G) \supset C^2(G) \supset \dots \supset C^n(G) = \{e\}$$ such that

- $(G, C^k(G)) \subset C^{k+1}(G)$
- $C^k(G)/C^{k+1}(G)$ is abelian $\left(\text{since } (C^k(G), C^k(G)) \subset (G, C^k(G)) = C^{k+1}(G)\right)$

Remarks. • $C^2(G) = (G, G) = D^1(G)$

• $C^{n+1}(G) \subset C^n(G)$ because $\forall\ g \in G,\ a \in C^n(G)$

typical element of the set generating $C^{n+1}(G)$ $\longrightarrow$ $\underbrace{g a g^{-1}}_{\in C^n(G)} a^{-1} \in C^n(G)$

as $C^n(G)$ is normal

(Homework exercise) • $(C^m(G), C^n(G)) \subset C^{m+n}(G)$

$\Rightarrow D^\ell(G) \subset C^{2^\ell}(G)$. Hence, nilpotent $\Rightarrow$ solvable.

We have the following analogue of Theorem 10.1 (p.2)

(10.6) Theorem. G is nilpotent if and only if it has a composition series $\Sigma : G = G_0 \supset G_1 \supset \dots \supset G_m = \{e\}$ such that

[$gr_j^{\Sigma}(G) = G_j / G_{j+1}$ is abelian $\forall\ 0 \le j \le m-1$]

- $(G, G_j) \subset G_{j+1} \quad \forall\ 0 \le j \le m-1$

this follows from $(G, G_j) \subset G_{j+1}$, since then $(G_j, G_j) \subset (G, G_j) \subset G_{j+1} \Rightarrow G_j / G_{j+1}$ is abelian.

Proof. ($\Rightarrow$) take G_j to be $C^{j+1}(G)$.

($\Leftarrow$) Again we claim that $C^{j+1}(G) \subset G_j$ $(0 \le j \le m)$.

Indeed $C^1(G) = G = G_0$ and $C^{j+1}(G) = (G, C^j(G))$ is a subset of $(G, G_{j-1}) \subset G_j$. Thus $C^{m+1}(G) = \{e\}$
(induction argument)

and hence G is nilpotent.

(10.7) Prop. (i) Subgroups and quotients of nilpotent groups are nilpotent

[use same argument as in the ($\Rightarrow$) proof of Prop. (10.3)]

(ii) G is nilpotent if and only if there is a subgroup $A \subset Z(G)$ s.t. G/A is nilpotent.

$\uparrow$ center of $G = \{x \in G : gx = xg \ \forall g \in G\}$

Proof. We only need to prove ($\Leftarrow$) implication, as the forward one follows from (i). Now let $\pi : G \to G/A$ be the natural surjection and let n be such that $C^n(G/A) = \{e\}$.

Then $\pi(C^n(G)) = C^n(G/A) \Rightarrow C^n(G) \subset A$. But $A \subset Z(G)$, so $C^{n+1}(G) \subset (G, A) = \{e\}$. $\square$

(10.8) Remark. The conclusion of Prop 10.7 (ii) above would be false if we drop the assumption of A being in the center.

In other words, if $1 \to G_1 \to G_2 \to G_3 \to 1$ is an exact sequence; G_1 & G_3 nilpotent <u>does not imply</u> G_2 is.

eg.

$$G_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : \begin{array}{l} a, d \in \mathbb{C}^\times \\ b \in \mathbb{C} \end{array} \right\}$$

$$\cup$$

$$G_3 \simeq \mathbb{C}^\times \times \mathbb{C}^\times \Leftarrow \quad G_1 = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\}$$

G_1 and G_3 are nilpotent; G_2 is solvable but not nilpotent (Homework exercise).

(10.9) Cor. Every p-group is nilpotent.

Proof. Let G be a group, $|G| = p^r$. We will prove this corollary by induction on r.

$(r=1)$ $G \simeq \mathbb{Z}/p\mathbb{Z}$ $\quad C^2(G) = \{e\}$ so G is nilpotent.

In general $\exists$ $H \simeq \mathbb{Z}/p\mathbb{Z}$, $H \subset Z(G)$ (see §7.3 p.4)

By induction hypothesis G/H is nilpotent and hence by Prop 10.7 (ii) of previous page, G is nilpotent $\square$

(10.10) In fact the only finite nilpotent groups are direct products of p-groups. The proof of this fact is a homework exercise, and depends on the following

Lemma. Let G be a nilpotent group and H be a proper subgroup. Then $H \subsetneq N_G(H)$.

Proof. As G is nilpotent, there is a composition series

$$G = G_0 \supset \ldots \supset G_n = \{e\} \quad \text{s.t. } (G, G_j) \subset G_{j+1}$$

Problem #5 of Set 4 $\Rightarrow$ $G_{j+1}H$ is normal in $G_j H$

so we get:

$$G = G_0 \cdot H \supset \ \ldots \ \supset G_{n-1} H \supset G_n H = H$$

Let k be largest subscript for which $G_k \cdot H \supsetneq H = G_{k+1} H$

Then $H \underset{\neq}{\overset{\text{normal}}{\triangleleft}} G_k \cdot H$ and hence $N_G(H) \supset G_k \cdot H$ implying

$N_G(H) \supsetneq H$. $\square$