Lecture 12

(12.0) Recall : last time we proved the following results about S_n :

- S_n is generated by transpositions ; even simple transpositions $:= \{ s_i = (i \ i+1) : 1 \le i \le n-1 \}$
- $\sigma \in S_n$. Assume $\sigma = \tau_1 \cdots \tau_p = \tau_1' \cdots \tau_q'$ where $\tau_1, \dots, \tau_p ; \tau_1', \dots, \tau_q'$ are transpositions, then $p \equiv q \mod 2$.

$\Big(\Rightarrow \exists !$ group hom. $\text{sign} : S_n \longrightarrow \{\pm 1\}$ s.t. $\text{sign}(\text{transposition}) = -1 \Big)$

- $A_n = \text{Ker}(\text{sign})$ is simple $\forall\, n \ge 5$.

(12.1) Example of $d \,\big|\, |G|$ but there is no subgroup $H < G$ s.t. $|H| = d$.

Take $G = A_5$. $|G| = \dfrac{5!}{2} = 60$. Let $d = 30$.

If $\exists\, \underset{\text{subgr}}{H < G}$ s.t. $|H| = 30$, i.e. $(G : H) = 2$ then H is normal (general fact / exercise : $(G:H) = 2$ implies H is normal) contradicting simplicity of A_5.

(12.2) Some relations among $s_1, \dots, s_{n-1} \in S_n$.

- $s_i^2 = 1 \quad \forall\ 1 \leq i \leq n-1$ (recall: $s_i = (i\ i{+}1)$)

- if i and j are far apart ($i < i+1 < j$)

$$s_i s_j = s_j s_i$$ (because disjoint cycles commute see Cor 11.1 (i) p.2)

- $\forall\ i;\ 1 \leq i \leq n-2$,

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$ (both $= (i\ i{+}2)$)

(12.3) A presentation of S_n.

Theorem. S_n admits the following presentation (see §1.6 of Lecture 1 p. 6)

$$\left\langle s_1, \dots, s_{n-1}\ ;\ \underset{(1 \leq i \leq n-1)}{s_i^2 = 1},\ \underset{(|i-j|>1)}{s_i s_j = s_j s_i},\ \underset{(1 \leq i \leq n-2)}{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}} \right\rangle$$

There are many proofs of this theorem. The one given below has a vast generalization. A more elementary one will be given in the homework.

(12.4) Lemma [Exchange Property]

Let $\pi \in S_n$, $k \in \{1, \dots, n-1\}$ be such that $\pi(k) > \pi(k+1)$.

Assume $\pi = s_{i_1} \cdots s_{i_\ell}$ for $i_1, \dots, i_\ell \in \{1, \dots, n-1\}$

Then $\exists\ j;\ 1 \leq j \leq \ell$ s.t.

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} s_k$$

Proof. Let us list

$$\pi = \pi_1 \ ; \ \pi_2 = s_{i_2} \cdots s_{i_\ell} \ ; \ \cdots \ ; \ \pi_\ell = s_{i_\ell} \ ; \ \pi_{\ell+1} = e$$

here $\pi_1(k) > \pi_1(k+1)$

here $\pi_{\ell+1}(k) < \pi_{\ell+1}(k+1)$

$\Rightarrow \exists\ j,\ 1 \leq j \leq \ell$ s.t. $\pi_{j+1}(k) < \pi_{j+1}(k+1)$

but $\pi_j(k) > \pi_j(k+1)$

$\sigma = \pi_{j+1}$. $\sigma(k) < \sigma(k+1)$

(for notational convenience) & $s_{i_j} \sigma(k) > s_{i_j} \sigma(k+1)$

But $s_{i_j} = (i_j \ \ i_j+1)$. So we get $\sigma(k) = i_j < i_j+1 = \sigma(k+1)$

$\Rightarrow s_{i_j} \sigma s_k = \sigma$ (Proof: for $\ell \notin \{k, k+1\}$

$$s_{i_j} \sigma s_K(\ell) = s_{i_j} \sigma(\ell) = \sigma(\ell) \quad \left[\text{as } \sigma(t) \in \{i_j, i_j+1\} \iff t \in \{K, K+1\} \right.$$

$$s_{i_j} \sigma s_K(K) = s_{i_j} \sigma(K+1) = s_{i_j}(i_j+1) = i_j = \sigma(K)$$

$$s_{i_j} \sigma s_K(K+1) = s_{i_j} \sigma(K) = s_{i_j}(i_j) = i_j+1 = \sigma(K+1)$$

$\square$