Lecture 13

(13.0) Recall: we proved the following results about S_n

- S_n is generated by $s_i = (i \ \ i+1)$ $(1 \leq i \leq n-1)$
- These elements $\{s_i\}_{1 \leq i \leq n-1}$ satisfy the following

list of relations:

$$s_i^2 = 1 \quad (\forall\ 1 \leq i \leq n-1) \qquad s_i s_j = s_j s_i \quad (\text{if } |i-j|>1) \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (\forall\ 1 \leq i \leq n-2)$$

[Lemma 12.4]
Exchange property: let $\pi \in S_n$ and $k \in \{1, \dots, n-1\}$ be such that $\pi(k) > \pi(k+1)$. Assume

$$\pi = s_{i_1} \cdots s_{i_\ell} \qquad (i_1, \dots, i_\ell \in \{1, \dots, n-1\})$$

<u>NOT necessarily</u> distinct

Then $\exists\ 1 \leq j \leq \ell$ s.t.

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} s_k$$

[Hence, $\pi = s_{i_1} \cdots s_{i_{j-1}} (s_{i_j} \cdots s_{i_\ell})$

$= s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_\ell} s_k$]

(13.1) A generalization. - Coxeter groups.

Let W be a group and $S \subset W$ be a subset satisfying

- $e \notin S$; $s^2 = e \ \forall s \in S$
- W is generated by S.

Definition. For $w \in W$, length of w, denoted by $\ell(w)$, is the smallest number ℓ such that w can be written as a product of ℓ elements of S.

$$w = s_1 \cdots s_\ell \qquad (\ell = \ell(w))$$

a reduced expression of w.

Lemma. Let $w \in W$ and $s \in S$.

(i) $\ell(ws) < \ell(w) \implies \ell(ws) = \ell(w) - 1$

(ii) $\ell(ws) > \ell(w) \implies \ell(ws) = \ell(w) + 1$

Proof. (i) Let $\ell(ws) = k < \ell = \ell(w)$

Then, if $ws = s_1' \cdots s_k'$ is a reduced exp.

then $w = s_1' \cdots s_k' s$ is an expression for w

$\implies \ell \leq k+1$. But $k < \ell \implies \ell = k+1$.

(ii) is proved analogously. $\square$

(13.2) Definition. We say W (together with S) is a Coxeter group if the <u>exchange property</u> holds:

Let $w \in W$, $s \in S$ such that $\ell(ws) \leq \ell(w)$. Let $w = s_1 \cdots s_\ell$ be an expression of w. Then $\exists\ 1 \leq j \leq \ell$ s.t.

$$s_j s_{j+1} \cdots s_\ell = s_{j+1} \cdots s_\ell \cdot s$$

[and hence $w = s_1 \cdots s_{j-1} s_{j+1} \cdots s_\ell \cdot s$]
and $\ell(ws) < \ell(w)$

Homework: for $W = S_n$ and the set $\{s_1, \ldots, s_{n-1}\}$

$$\ell(\pi s_k) < \ell(\pi) \iff \pi(k) > \pi(k+1)$$

(13.3) Again let W be a Coxeter group. For every $s, s' \in S$, let $m(s, s')$ be the order of $s \cdot s' \in W$ (could very well be infinity). [$m(s,s) = 1$]

Theorem. W has a presentation:

Generators: $s \in S$

Relations: $\forall\ s, s' \in S$

$$\underbrace{s s' s s' \cdots}_{m(s,s')\text{-terms}} = \underbrace{s' s s' s \cdots}_{m(s,s')\text{ terms}}$$

Proof. We are given a group G and a set map $\varphi : S \to G$ such that $\forall\, s, s' \in S$

$$(*) \qquad \varphi(s)\,\varphi(s')\,\varphi(s)\,\varphi(s')\cdots = \varphi(s')\,\varphi(s)\,\varphi(s')\,\varphi(s)\cdots$$

[$m(s,s')$ terms on both sides]

To prove: φ extends to a unique gp. hom. $W \xrightarrow{\phi} G$.

Let $w \in W$ and choose a reduced expression $w = s_1 \cdots s_\ell$. Define $\phi(w) := \varphi(s_1)\cdots\varphi(s_\ell)$
$(\ell = \ell(w))$ $\qquad (= e$ if $\ell = 0)$

Claim: $\phi(w)$ is well-defined (i.e. does not depend on the choice of the reduced expression)

Proof of the claim: Let $w = s_1 \cdots s_\ell = s_1' \cdots s_\ell'$ be two reduced expressions.

We need to prove that $\varphi(s_1)\cdots\varphi(s_\ell) = \varphi(s_1')\cdots\varphi(s_\ell')$

If $\ell = 0$, $w = e$ and there is nothing to prove.
Assume the claim has been proven for all w of length $< \ell$.

Take $w \in W$ of length ℓ and two reduced expressions

$$w = s_1 \cdots s_\ell \quad \rightsquigarrow \quad \phi_1(w) = \varphi(s_1) \cdots \varphi(s_\ell)$$

$$= s_1' \cdots s_\ell' \qquad\qquad \phi_2(w) = \varphi(s_1') \cdots \varphi(s_\ell')$$

<u>Note</u> : if $s_\ell = s_\ell'$, then we are done by induction :
(or $s_1 = s_1'$)

$$\phi_1(w) = [\varphi(s_1) \cdots \varphi(s_{\ell-1})]\, \varphi(s_\ell) \overset{\text{by induction}}{=} [\varphi(s_1') \cdots \varphi(s_{\ell-1}')]\, \varphi(s_\ell)$$

$$\underset{\text{as } s_\ell = s_\ell'}{=} \varphi(s_1') \cdots \varphi(s_{\ell-1}')\, \varphi(s_\ell') = \phi_2(w)$$

Otherwise we use Exchange Property

$\ell(w s_\ell') < \ell(w)$ $\implies$ $\exists\, 1 \leq j \leq \ell$ such that
$w = s_1 \cdots s_\ell$

$$s_j \cdots s_\ell = s_{j+1} \cdots s_\ell s_\ell'$$

so $w = s_1 \cdots s_{j-1} s_{j+1} \cdots s_\ell s_\ell'$

if $j \neq 1$ here, we found a reduced expression of w whose first term is s_1 and last term is s_ℓ'. By the <u>Note</u> above

$$\phi_3(w) = \varphi(s_1) \cdots \varphi(s_{j-1})\, \varphi(s_{j+1}) \cdots \varphi(s_\ell)\, \varphi(s_\ell')$$

$$= \phi_1(w) \quad (\text{same } 1^{st} \text{ term})$$

$$= \phi_2(w) \quad (\text{same last term})$$

→ let us assume $j = 1$ in all future applications of E.P.

$$s_1 \cdots s_\ell \xrightarrow{\substack{\text{[EP] with} \\ \ell(ws_\ell') < \ell(w)}} s_2 \cdots s_\ell s_\ell' \rightsquigarrow s_3 \cdots s_\ell s_\ell' s_\ell \quad \cdots$$

$$s_1' \cdots s_\ell' \xrightarrow[\substack{\text{[EP] with} \\ \ell(ws_\ell) < \ell(w)}]{} s_2' \cdots s_\ell' s_\ell \rightsquigarrow s_3' \cdots s_\ell' s_\ell s_\ell'$$

- - - means same last term and hence same element is obtained in G.

Thus we will get two reduced expressions
$$\cancel{s_\ell} \cdots s_\ell' s_\ell s_\ell' s_\ell$$
$$\cdots s_\ell' s_\ell s_\ell'$$

# of terms has to be $m(s_\ell, s_\ell')$ otherwise the expression will not be reduced.

$$\text{So} \quad \phi_1(w) \underset{\substack{\uparrow \\ \text{same last} \\ \text{term}}}{=} \cdots \varphi(s_\ell')\varphi(s_\ell) \underset{\substack{\uparrow \\ (*) \text{ of p. 2}}}{=} \cdots \varphi(s_\ell)\varphi(s_\ell') \underset{\substack{\uparrow \\ \text{same last} \\ \text{term}}}{=} \phi_2(w)$$

— end of the proof of the claim.

Now we need to check $\phi : W \to G$ is a group hom.

It is enough to show that
$$\phi(ws) = \phi(w)\varphi(s) \quad \forall\, w \in W,\ s \in S.$$

- if $\ell(ws) > \ell(w)$. If $w = s_1 \cdots s_\ell$ is a red. exp. of w then $s_1 \cdots s_\ell s$ is a red. exp. of ws

$$\Rightarrow \quad \phi(ws) = \varphi(s_1) \cdots \varphi(s_\ell)\varphi(s) = \phi(w)\varphi(s).$$

- if $\ell(ws) < \ell(w)$. If $w' = ws = s_1' \cdots s_k'$ is a reduced expression for w', then $s_1' \cdots s_k' \cdot s = w$ is a red. exp

$$\Rightarrow \quad \phi(w) = \phi(w')\varphi(s) \underset{(\varphi(s)^2 = 1)}{\Rightarrow} \phi(w)\varphi(s) = \phi(w')$$

$\square$