Lecture 14

(14.0) In this lecture we review the basic operations on vector spaces (over $\mathbb{C}$).

Direct Sums, Tensor product, Hom, Dual vector space, Symmetric and alternating (or exterior) product

Recall: a vector space V (for us, always over $\mathbb{C}$) is a set V together with ~~two~~ 3 operations; and an element $0 \in V$

$$+ : V \times V \to V \qquad\qquad V \to V$$
$$(v_1, v_2) \mapsto v_1 + v_2 \qquad\qquad v \mapsto -v$$

$\Big\}$ abelian group with 0 as the identity elt.

$$\mathbb{C} \times V \to V \quad \text{(scalar multiplication)}$$
$$(z, v) \mapsto zv$$

such that

$$z(v_1 + v_2) = zv_1 + zv_2$$
$$z_1(z_2 v) = (z_1 z_2) v$$
$$(z_1 + z_2) v = z_1 v + z_2 v$$

$\forall\ z, z_1, z_2 \in \mathbb{C}$, $v, v_1, v_2 \in V$

(14.1) Linear maps: A ($\mathbb{C}$-) linear map from a vector space V to a vector space W is a group hom. $f : V \to W$ s.t.

$$f(z \cdot v) = z \cdot f(v) \quad \forall\ z \in \mathbb{C},\ v \in V.$$

$\mathrm{Hom}_{\mathbb{C}}(V, W)$ = set of all linear maps from V to W

Vector space structure on $\mathrm{Hom}_{\mathbb{C}}(V, W)$

- $\forall f_1, f_2 \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, $f_1 + f_2$ is defined to be

$$(f_1 + f_2)(v) = f_1(v) + f_2(v) \quad \forall v \in V$$

- Zero map $\quad 0 \in \mathrm{Hom}_{\mathbb{C}}(V, W) \qquad 0 : v \mapsto 0 \ \forall v \in V.$

- Scalar multiplication : $\quad \forall z \in \mathbb{C},\ f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$

$$(zf)(v) = z f(v).$$

Note: We never really used the vector space structure of V in the definition of the vector space structure on $\mathrm{Hom}_{\mathbb{C}}(V, W)$. The same would work to make $\mathrm{Hom}_{set}(X, W)$ a vector space where X is any set and W is a vector space.

Remark. If V and W are finite-dimensional of dimensions say m and n; $\mathrm{Hom}_{\mathbb{C}}(V, W)$ can be identified with the set of $n \times m$ matrices with entries from $\mathbb{C}$. This involves choose a basis $\{v_i\}_{1 \leq i \leq m}$ of V and a basis $\{w_j\}_{1 \leq j \leq n}$ of W. Then $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ can be expressed as

$$f(v_i) = \cancel{\sum_{j=1}^{n} a_{ij} w_j} = \sum_{j=1}^{n} a_{ji} w_j$$

which gives the matrix $\quad (a_{ji})_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} \in M_{n \times m}(\mathbb{C})$

(14.2) Direct Sum. Let V_1 and V_2 be two vector spaces. $V_1 \oplus V_2$ denotes the vector space which is same as the Cartesian product $V_1 \times V_2$ as a set with the following structure

- $(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2')$
- $z(v_1, v_2) = (zv_1, zv_2)$

If $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are two linear maps $f_1 \oplus f_2 \in \mathrm{Hom}_{\mathbb{C}}(V_1 \oplus V_2, W_1 \oplus W_2)$ is defined by

$$(v_1, v_2) \longmapsto (f_1(v_1), f_2(v_2))$$

In terms of matrices $X_1 \oplus X_2$ is the block matrix

$$\begin{array}{c c} & \begin{array}{cc} \overbrace{}^{m_1} & \overbrace{}^{m_2} \end{array} \\ \begin{array}{c} n_1 \{ \\ n_2 \{ \end{array} & \left[\begin{array}{c|c} X_1 & 0 \\ \hline 0 & X_2 \end{array}\right]_{(n_1+n_2) \times (m_1+m_2)} \end{array}$$

$m_1, m_2 = \dim(V_1), \dim(V_2)$
$n_1, n_2 = \dim(W_1), \dim(W_2)$
resp.

Note; if $\{v_i^{(1)}\}_{1 \le i \le m_1}$ is a basis of V_1 and $\{v_j^{(2)}\}_{1 \le j \le m_2}$ is a basis of V_2, then $\{(v_i^{(1)}, 0), (0, v_j^{(2)})\}_{\substack{1 \le i \le m_1 \\ 1 \le j \le m_2}}$ is a basis of $V_1 \oplus V_2$.

(14.3) Dual. Let V be a vector space. The dual of V, denoted by V^*, is defined as $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$

1-dim'l vector space

If V is finite dimensional, then $\dim V^* = \dim V$.
Let $\{v_i\}_{1 \le i \le m}$ be a basis of V. Define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Then $\{v_i^*\}_{1 \le i \le m}$ is a basis of V^*. [Note: not true for infinite-dimensional vector spaces]

If $f : V \to W$ is a linear map, $f^* : W^* \to V^*$ is defined by

$$f^*(\xi)(v) = \xi(f(v)) \quad \forall\, \xi \in W^*,\ v \in V.$$

In other words, $f^*(\xi) = \xi \circ f : V \xrightarrow{f} W \xrightarrow{\xi} \mathbb{C}$

In terms of matrices f^* becomes transpose of f.

(14.4) Bilinear maps and tensor product

Let V_1, V_2, W be three vector spaces. A bilinear map $f : V_1 \times V_2 \to W$ is a set map which is linear in each coordinate: i.e., $\forall\, v_1 \in V_1$;

$$\begin{aligned} V_2 &\to W \\ v &\mapsto f(v_1, v) \end{aligned} \quad \in \mathrm{Hom}_{\mathbb{C}}(V_2, W)$$

& $\forall\, v_2 \in V_2$;

$$\begin{aligned} V_1 &\to W \\ v' &\mapsto f(v', v_2) \end{aligned} \quad \in \mathrm{Hom}_{\mathbb{C}}(V_1, W)$$

$V_1 \underset{\mathbb{C}}{\otimes} V_2$ is a vector space together with a bilinear map

$$\begin{array}{ccc} V_1 \times V_2 & \longrightarrow & V_1 \otimes V_2 \\ (v_1, v_2) & \longmapsto & v_1 \otimes v_2 \end{array}$$

such that for any vector space W and bilinear map $f : V_1 \times V_2 \to W$, $\exists!\ \tilde{f} : V_1 \otimes V_2 \to W$ (linear map) making the following diagram commute

$$\begin{tikzcd} V_1 \times V_2 \arrow[r] \arrow[dr, "f"'] & V_1 \otimes V_2 \arrow[d, dashed, "\exists!\ \tilde{f}"] \\ & W \end{tikzcd}$$

Construction: $V_1 \otimes V_2$ is a vector spanned spanned by $\cancel{\{v_1 \otimes v_2}$ $\{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$ quotiented by the subspace spanned by

- $(z_1 v_1 + z_1' v_1', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)$
- $(v_1, z_2 v_2 + z_2' v_2') - z_2 (v_1, v_2) - z_2' (v_1, v_2')$

$\forall\ z_1, z_2, z_1', z_2' \in \mathbb{C}$; $v_1, v_1' \in V_1$; $v_2, v_2' \in V_2$.

If $f_1 : V_1 \to W_1$ & $f_2 : V_2 \to W_2$ are $\mathbb{C}$-linear, then

$$\begin{tikzcd} (f_1, f_2) : V_1 \times V_2 \arrow[r] \arrow[dr, dashed, "\text{bilinear}"'] & W_1 \times W_2 \arrow[d] \\ & W_1 \otimes W_2 \end{tikzcd}$$

$$\rightsquigarrow \quad f_1 \otimes f_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

Remarks. (i) If $\{v_i^{(1)}\}_{1 \le i \le m_1}$ is a basis of V_1 and $\{v_j^{(2)}\}_{1 \le j \le m_2}$ is a basis of V_2; then $\{v_i^{(1)} \otimes v_j^{(2)}\}$ is a basis of $V_1 \otimes V_2$ [in particular, $\dim(V_1 \otimes V_2) = \dim V_1 \cdot \dim V_2$].

(ii) Let $n_1 = \dim W_1$ and $n_2 = \dim W_2$. Assume $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are identified with matrices $X_1 \in M_{n_1 \times m_1}(\mathbb{C})$, $X_2 \in M_{n_2 \times m_2}(\mathbb{C})$. Then $f_1 \otimes f_2$ gets identified with a matrix $X_1 \otimes X_2 \in M_{n_1 n_2 \times m_1 m_2}(\mathbb{C})$:

$$X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1,m_1} \\ \vdots & \ddots & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,m_1} \end{bmatrix} \qquad X_2 = \begin{bmatrix} b_{11} & \cdots & b_{1,m_2} \\ \vdots & \ddots & \vdots \\ b_{n_2,1} & \cdots & b_{n_2,m_2} \end{bmatrix}$$

$$X_1 \otimes X_2 = \begin{bmatrix} a_{11} X_2 & & a_{1,m_1} X_2 \\ \boxed{a_{21} X_2} & \cdots & a_{2,m_1} X_2 \\ \vdots & & \vdots \\ a_{n_1,1} X_2 & & a_{n_1,m_1} X_2 \end{bmatrix}$$

$n_2 \times m_2$ matrix

in (ordered) basis $\cancel{\{v_1^{(1)} \otimes w_1^{(1)}, v_1^{(1)} \otimes w_2^{(1)}}$

$$\{ v_1^{(1)} \otimes v_1^{(2)}, \ldots, v_1^{(1)} \otimes v_{m_2}^{(2)}, v_2^{(1)} \otimes v_1^{(2)}, \ldots, v_2^{(1)} \otimes v_{m_2}^{(2)}, \ldots \}$$

(14.5) Hom - Tensor adjointness.

Prop. There is a natural map $V^* \otimes W \xrightarrow{\phi} \mathrm{Hom}(V,W)$.

$$\phi(\xi \otimes w) : v \longmapsto \xi(v) \cdot w$$

Moreover ϕ is an isomorphism, if V [and, W] are finite-dim'l.
↑ not needed
(In general, ϕ is always injective).

Proof: Define
$$\varphi : V^* \times W \longrightarrow \mathrm{Hom}(V,W)$$
$$(\xi, w) \longmapsto \{ v \mapsto \xi(v) w \}$$

Easy check: φ is bilinear. Hence yields a linear map
$$\phi : V^* \otimes W \to \mathrm{Hom}(V,W) \text{ s.t. } \phi(\xi \otimes w) = \varphi(\xi, w).$$

ϕ is injective: Let $\alpha \in V^* \otimes W$ be such that $\phi(\alpha) = 0$.

$$\alpha = \sum_{j=1}^{N} \xi_j \otimes w_j$$ and we can assume $w_1, \dots, w_N$ are linearly independent.

Then $\forall v \in V$, $\sum_{j=1}^{N} \xi_j(v) \cdot w_j = \phi(\alpha)(v) = 0$

$\Rightarrow \xi_j(v) = 0 \quad \forall \, 1 \leq j \leq N$ (as $w_1, \dots, w_N$ are linearly independent)

So $\xi_j(v) = 0 \ \forall v \in V \Rightarrow \xi_j = 0 \quad (\forall \, 1 \leq j \leq N)$

$\Rightarrow \alpha = 0$.

If V is finite-dim'l, we can prove ϕ is also surjective

Let $\{v_1, \dots, v_m\}$ be a basis of V. Given $f \in \mathrm{Hom}(V, W)$ let $w_i = f(v_i)$ $(1 \leq i \leq m)$. Then $\alpha = \sum_{i=1}^{m} v_i^* \otimes w_i \in V^* \otimes W$ is such that $\phi(\alpha) = f$.

□

Cor. Let V be a finite-dim'l vector space. Choose a basis $\{v_i\}_{1 \leq i \leq m}$ of V and let $\{v_i^*\}_{1 \leq i \leq m}$ be the basis dual to it. Then $\sum_{i=1}^{n} v_i^* \otimes v_i$ is independent of the choice of basis.

Proof. If $\{\tilde{v}_i\}_{1 \leq i \leq m}$ is another basis of V, then

$$\left.\begin{aligned} \alpha &= \sum_{i=1}^{m} v_i^* \otimes v_i \\ \beta &= \sum_{i=1}^{m} \tilde{v}_i^* \otimes \tilde{v}_i \end{aligned}\right\} \in V^* \otimes V \quad \text{and} \quad \phi(\alpha) = \phi(\beta) = \mathrm{Id}_V \in \mathrm{Hom}(V, V)$$

$\Rightarrow \quad \alpha = \beta$

□

(14.6) Bilinear forms. A bilinear form on V is a bilinear map from $V \times V$ to $\mathbb{C}$. Equivalently, it is a linear map $V \otimes V \to \mathbb{C}$, i.e. an element of $(V \otimes V)^*$.

$B : V \times V \to \mathbb{C}$ bilinear is said to be non-degenerate if

$$B(v_1, v_2) = 0 \;\; \forall\, v_2 \;\Rightarrow\; v_1 = 0$$
$$B(v_1, v_2) = 0 \;\; \forall\, v_1 \;\Rightarrow\; v_2 = 0 .$$

(14.7) $\mathrm{Sym}^k(V)$ and $\Lambda^k V$.

Inductively we can define $V^{\otimes k} = \underbrace{V \otimes V \otimes \dots \otimes V}_{k\text{-terms}}$ $(\forall k \geq 2)$

$$\mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\{ \begin{array}{l} \text{Subspace spanned by} \\ v_1 \otimes \dots \otimes v_k - v_1 \otimes \dots \otimes v_{i-1} \otimes (v_{i+1} \otimes v_i) \otimes v_{i+2} \otimes \dots \otimes v_k \\ \qquad\qquad\qquad (1 \leq i \leq k-1) \\ v_1, \dots, v_k \in V \end{array} \right\}$$

(k^{th} symmetric product of V)

$$\Lambda^k(V) = V^{\otimes k} \Big/ \left\{ \begin{array}{l} \text{Subspace spanned by} \\ v_1 \otimes \dots \otimes v_k + v_1 \otimes \dots \otimes v_{i-1} \otimes [v_{i+1} \otimes v_i] \otimes v_{i+2} \otimes \dots \otimes v_k \\ \qquad\qquad\qquad (1 \leq i \leq k-1) \\ v_1, \dots, v_k \in V \end{array} \right\}$$

(k^{th} exterior product of V)

A typical summand of a term in $\mathrm{Sym}^k(V)$ is written as $v_1 \dots v_k$ (just a monomial- order of terms is immaterial); while for $\Lambda^k(V)$, it is written as $v_1 \wedge \dots \wedge v_k$ (order matters!)

- $v_1 \wedge \dots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$. [flipping factors introduces a $-$ve sign]
- $v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = \mathrm{sign}(\sigma)(v_1 \wedge \dots \wedge v_k)$ $\forall \sigma \in S_k$.

If $\{v_1, \dots, v_m\}$ is a basis of V, then

$$\left\{ v_1^{r_1} \cdots v_m^{r_m} \;:\; \begin{array}{l} r_1, \ldots, r_m \geq 0 \\ r_1 + \ldots + r_m = k \end{array} \right\} \text{ is a basis of } \mathrm{Sym}^k(V)$$

$$\left\{ v_{i_1} \wedge \ldots \wedge v_{i_k} \;:\; 1 \leq i_1 < i_2 \cdots < i_k \leq m \right\} \text{ is a basis of } \Lambda^k(V).$$

[In particular, $\dim(\Lambda^k V) = \binom{m}{k}$ $(= 0 \text{ if } k > m)$]

Convention : $\mathrm{Sym}^0(V) = \Lambda^0 V = \mathbb{C}$

$\mathrm{Sym}^1(V) = \Lambda^1(V) = V$

Lemma $V \otimes V \simeq \mathrm{Sym}^2(V) \oplus \Lambda^2(V)$

Proof. As $\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$ are defined as quotients of $V \otimes V$, we get a natural linear map

$$\begin{aligned} V \otimes V &\longrightarrow \mathrm{Sym}^2(V) \oplus \Lambda^2(V) \\ v \otimes v' &\longmapsto (v.v', \; v \wedge v') \end{aligned}$$

For later.