Lecture 15

(15.0) Recall: last time we introduced the following operations on vector spaces (over $\mathbb{C}$): direct sum $\oplus$, tensor product, symmetric and exterior products.

$\mathrm{Hom}_{\mathbb{C}}(V,W)$ = vector space of linear maps from V to W.

$f \in \mathrm{Hom}_{\mathbb{C}}(V,W) \rightsquigarrow$ (as usual) $\mathrm{Ker}(f) = \{v \in V : f(v) = 0\}$ - subspace of V

$$\mathrm{Im}(f) = \{f(v) : v \in V\} \subset W \quad \text{subspace}$$

(15.1) Definition. Let G be a group and V a vector space. We say V is a representation of G if we have a group homomorphism $\rho : G \longrightarrow GL(V) \leftarrow$ the group of linear isomorphisms $V \to V$.

I usually write it as $G \circlearrowright V$ (read: G acts on V via linear automorphisms) and suppress ρ, if no confusion is possible.

Example: Let X be a set and assume $G \circlearrowright X$.

Recall that this means that we have a group hom

$$\tau : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X).$$

Take $V = \mathrm{Fun}(X) :=$ all $\mathbb{C}$-valued functions $X \to \mathbb{C}$.
(see note on page 2 of Lecture 14)

$G \circlearrowright V$ via $G \xrightarrow{\rho} GL(V)$ defined as

$$[\rho(g)(f)](x) = f\left(\tau(g^{-1})(x)\right) \quad \forall \begin{array}{l} f \in \mathrm{Fun}(X) = V \\ g \in G \\ x \in X \end{array}$$

Check: ρ is a group hom. This verification will explain the appearance of g^{-1} in the formula of ρ.

Example of example. $G = S_n \circlearrowright X = \{1, \dots, n\}$.

$V = \mathrm{Fun}(X) \simeq \mathbb{C}^n$ by taking the basis of "step functions"
$e_i \in V$ is the function $e_i(j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$

$G \circlearrowright V$ by $(\sigma \cdot e_i)(j) = e_i(\sigma^{-1}(j)) = \begin{cases} 1 & \text{if } i = \sigma^{-1}(j) \\ 0 & \text{o/w} \end{cases}$

$\Rightarrow \quad \sigma \cdot e_i = e_{\sigma(i)}$

(15.2) Let $\rho : G \to GL(V)$ be a representation of G.
A subspace $V' \subset V$ is said to be <u>G-stable</u>; or a <u>subrepresentation</u>, if $\rho(g)(v) \in V' \quad \forall \begin{array}{l} g \in G \\ v \in V' \end{array}$.

Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of G (i.e., $\rho_1 : G \to GL(V_1)$ & $\rho_2 : G \to GL(V_2)$ are two group homomorphisms): A linear map $f \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is called a <u>G-intertwiner</u> or <u>G-linear</u> or <u>homomorphism of representations of G</u> if $\forall\, g \in G$ we have

$$f \circ \rho_1(g) = \rho_2(g) \circ f \quad \left(\text{i.e. } \begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \rho_1(g)} \downarrow & & \downarrow {\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array} \text{ commutes} \right).$$

$\mathrm{Hom}_G(V_1, V_2) :=$ vector space (subspace of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$) of all G-intertwiners.

Lemma. Let $f \in \mathrm{Hom}_G(V_1, V_2)$. Then $\mathrm{Ker}(f) \subset V_1$ and $\mathrm{Im}(f) \subset V_2$ are subrepresentations.

Proof is left as an easy exercise.

e.g. Consider the example of $S_n \circlearrowright \mathbb{C}^n$ from §15.1.

$V =$ Span of $e_1 + \dots + e_n \subset \mathbb{C}^n$ is a subrepn.

(15.3) The usual operations on vector spaces extend naturally to group representations. Let $\rho_1, \rho_2 : G \longrightarrow GL(V_1), GL(V_2)$ resp. be two representations of a group G.

- $V_1 \oplus V_2$ is a G-repn. via $\rho_\oplus : G \longrightarrow GL(V_1 \oplus V_2)$
$$\rho_\oplus(g) = \rho_1(g) \oplus \rho_2(g)$$
- $V_1 \otimes V_2$ is a G-repn. via $\rho_\otimes : G \longrightarrow GL(V_1 \otimes V_2)$
$$\rho_\otimes(g) = \rho_1(g) \otimes \rho_2(g)$$
- V_1^* is a G-repn. via $\rho_* : G \longrightarrow GL(V_1^*)$

$\forall\, g \in G,\ \xi \in V_1^*,\ v \in V_1$: $\left[\rho_*(g)(\xi)\right](v) = \xi(\rho_1(g^{-1})(v))$

- $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is a G-repn via $\rho_{\mathrm{Hom}} : G \longrightarrow GL\left(\mathrm{Hom}_{\mathbb{C}}(V_1,V_2)\right)$
$$\left[\rho_{\mathrm{Hom}}(g)(X)\right](v_1) = \rho_2(g)\left[X\left(\rho_1(g^{-1})(v_1)\right)\right]$$

HW: $\varphi : V_1^* \otimes V_2 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ of Prop. 14.5 (p.7)
is a G-intertwiner.

(15.4) A representation V of a group G is said to be

<u>Irreducible</u> : if $V' \subset V$ G-stable subspace, then either $V' = 0$ or $V' = V$

<u>Indecomposable</u> : if $V = V' \oplus V''$ as G-repns, then either $V' = V$ or $V'' = V$.

e.g. $S_n \circlearrowright \mathbb{C}^n$ is not irreducible.

Clearly irreducible implies indecomposable. The converse is not true. Consider $\mathbb{Z} \longrightarrow GL(\mathbb{C}^2)$. The resulting repn. is

$$n \longmapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

not irreducible since $\left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} : z \in \mathbb{C} \right\}$ is a $\mathbb{Z}$-stable subspace

However, it is indecomposable. Since on the contrary, we should be able to find a vector of the form $\alpha = \begin{bmatrix} a \\ 1 \end{bmatrix}$ $(a \in \mathbb{C})$ such that $\mathbb{C} \cdot \alpha$ is $\mathbb{Z}$-stable. That is $n \cdot \alpha$ is a scalar multiple of α.

i.e. $\begin{bmatrix} n+a \\ 1 \end{bmatrix}$ is proportional to $\begin{bmatrix} a \\ 1 \end{bmatrix}$ for every $n \in \mathbb{Z}$. This is absurd, proving that the representation $\mathbb{Z} \circlearrowright \mathbb{C}^2$ is indecomposable.