Lecture 16

(16.0) Recall : last time we introduced the notions of group representations, their direct sums, tensor products, duals and Hom's.

$$G \circlearrowright V_1, V_2 \rightsquigarrow G \circlearrowright V_1 \oplus V_2,\ V_1 \otimes V_2,\ V_1^*,\ \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$$

(16.1) An example : revisiting symmetric & alternating / exterior products.

Let V be a vector space and let $\ell \geq 2$. Consider the representation

$$S_\ell \circlearrowright V^{\otimes \ell} \quad : \quad \sigma \cdot (v_1 \otimes \cdots \otimes v_\ell) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(\ell)}.$$

Define

$$\mathcal{S}ym : V^{\otimes \ell} \longrightarrow V^{\otimes \ell}$$
$$\alpha \longmapsto \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \sigma \cdot \alpha$$

Prop. $\mathrm{Sym}^\ell(V) = \mathrm{Im}(\mathcal{S}ym) = (V^{\otimes \ell})^{S_\ell}$ $\left(:= \{ \alpha \in V^{\otimes \ell} : \sigma \cdot \alpha = \alpha\ \forall \sigma \in S_\ell \} \right)$

Proof. Recall that we defined (§14.7; p.9) $\mathrm{Sym}^\ell(V) = V^{\otimes \ell}/W$

W = subspace spanned by $\alpha - s_i \cdot \alpha$ $\quad \forall\ \alpha \in V^{\otimes \ell}$, $1 \leq i \leq \ell - 1$
$(s_i = (i\ i{+}1) \in S_\ell)$

$\underset{\substack{\uparrow \\ \text{as } \{s_1, \ldots, s_{\ell-1}\} \\ \text{generate } S_\ell}}{=}$ Subspace spanned by $\alpha - \sigma \cdot \alpha$ $\quad \forall\ \alpha \in V^{\otimes \ell},\ \sigma \in S_\ell$

$\underline{\mathrm{Im}(Sym) = (V^{\otimes \ell})^{S_\ell}}$: $\subseteq$ Let $\alpha = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \sigma . \beta \in \mathrm{Im}(Sym)$
$(\beta \in V^{\otimes \ell})$

Then $\forall\, \tau \in S_\ell$, we have $\tau . \alpha = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \tau\sigma . \beta = \frac{1}{\ell!} \sum_{\sigma' \in S_\ell} \sigma' . \beta = \alpha$

$\Rightarrow \alpha \in (V^{\otimes \ell})^{S_\ell}$.

Conversely if $\alpha \in (V^{\otimes \ell})^{S_\ell}$, then $Sym(\alpha) = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \underbrace{\sigma . \alpha}_{=\alpha} = \frac{1}{\ell!} . \ell! \, \alpha$

so $\alpha = Sym(\alpha) \in \mathrm{Im}(Sym)$.

Thus $V^{\otimes \ell} / \mathrm{Ker}(Sym) \simeq \mathrm{Im}(Sym) = (V^{\otimes \ell})^{S_\ell}$. To finish the proof of the proposition, it remains to show that $W = \mathrm{Ker}(Sym)$.

$W \subset \mathrm{Ker}(Sym)$: since $Sym(\alpha - \sigma . \alpha) = \frac{1}{\ell!} \left(\sum_{\tau \in S_\ell} \tau . \alpha - \sum_{\tau \in S_\ell} \tau\sigma . \alpha \right) = 0$.

and W is spanned by $\{\alpha - \sigma . \alpha : \alpha \in V^{\otimes \ell}, \sigma \in S_\ell\}$, we get that $W \subset \mathrm{Ker}(Sym)$.

$\mathrm{Ker}(Sym) \subset W$: Let $\alpha \in \mathrm{Ker}(Sym)$. Then $\alpha = \alpha - Sym(\alpha)$

$= \frac{1}{\ell!} \left\{ \sum_{\sigma \in S_\ell} \alpha - \sigma . \alpha \right\} \in W$.

$\square$

Analogously, we can prove that

$$\Lambda^{\ell}(V) \cong \{\alpha \in V^{\otimes \ell} : \sigma.\alpha = \mathrm{sign}(\sigma)\alpha \ \forall \sigma \in S_\ell\} \subset V^{\otimes \ell}$$

$$= \text{Image of } \left\{ \alpha \in V^{\otimes \ell} \longmapsto \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \mathrm{sign}(\sigma)\, \sigma.\alpha \right\}$$

(16.2) Averaging over group $\rightsquigarrow$ complete reducibility

Theorem [Maschke's Thm.] Let V be a representation, finite-dim'l, of a finite group G. Let $V_1 \subset V$ be a subrepresentation. Then there exists another subrepn. $V_2 \subset V$ such that $V \simeq V_1 \oplus V_2$.

Proof Write $V = V_1 \oplus W$ as vector space and define

$$f \in \mathrm{End}_{\mathbb{C}}(V) = \mathrm{Hom}_{\mathbb{C}}(V,V) \text{ by : } \begin{array}{l} f(v) = v \quad \forall\, v \in V_1 \\ f(w) = 0 \quad \forall\, w \in W. \end{array}$$

(in matrix form $f = \begin{array}{cc} \begin{array}{cc} V_1 & W \end{array} & \\ \left[\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} V_1 \\ W \end{array} \end{array}$)

Thus f is a projection onto $V_1 \subset V$. We can "average over G" to obtain

$$P := \frac{1}{|G|} \sum_{g \in G} g \cdot f \quad \in \mathrm{Hom}_{\mathbb{C}}(V,V)$$

(recall $g.X = g \circ X \circ g^{-1} \ \forall\, g \in G,\ X \in \mathrm{Hom}_{\mathbb{C}}(V,V)$)

- P is a G-intertwiner (clear ✓)
- P is a projection onto V_1 (i.e. $V_1 = \mathrm{Im}(P)$ and $P(v) = v \ \forall v \in V_1$)
  in particular, $P^2 = P$

since $\forall v \in V_1$, $P(v) = \frac{1}{|G|} \sum_{g \in G} g(f(g^{-1} \cdot v))$

$$= \frac{1}{|G|} \sum_{g \in G} g \cdot g^{-1} \cdot v = v.$$

since $f(v) = v$ $\forall v \in V_1$

Thus if we take $V_2 = \mathrm{Ker}(P) \subset V$ (a subrepn, since P is a G-intertwiner), we get $V \cong \mathrm{Ker}(P) \oplus \mathrm{Im}(P) = V_2 \oplus V_1$ as required
(Homework)

□

(16.3) Corollary (1) Let V be a finite-dimensional repn. of a finite group G. If V is indecomposable, then it is irreducible.

(2) Again let V be a finite-dimensional repn. of a finite group G. Then $V \cong V_1 \oplus \dots \oplus V_\ell$, where $V_1, \dots, V_\ell$ are irreducible, subrep-ns.

e.g. $G = S_n \circlearrowright \mathbb{C}^n \supset V = \mathbb{C}.(e_1 + \dots + e_n)$

Take $W = \{a_1 e_1 + \dots + a_n e_n : a_1, \dots, a_n \in \mathbb{C} ; a_1 + \dots + a_n = 0\}$

Then $\mathbb{C}^n = V \oplus W$ as S_n - representations.

(16.4) Schur's lemma. (i) Let V, W be two irreducible representations of a group G, and let $f \in \mathrm{Hom}_G(V, W)$. Then either $f = 0$ or f is an isomorphism.

(ii) Assume further that V is finite-dimensional and $f \in \mathrm{Hom}_G(V, V)$. Then $\exists\ \lambda \in \mathbb{C}$ s.t. $f = \lambda . \mathrm{Id}_V$.

Proof. (i) $f : V \longrightarrow W$. is G-intertwiner, so

$\mathrm{Ker}(f) \subset V$ is a subrep-n. $\Rightarrow \mathrm{Ker}(f) = 0$ or V

$\mathrm{Im}(f) \subset W$ is a subrep-n $\Rightarrow \mathrm{Im}(f) = 0$ or W

Hence either $f = 0$ or $(\mathrm{Ker}(f) = 0\ \&\ \mathrm{Im}(f) = W)$ i.e. f is an isomorphism $\square$

(ii) Let $f \in \mathrm{Hom}_G(V, V)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of f (i.e. $\exists\ v \in V$, $v \neq 0$ s.t. $f(v) = \lambda . v$)

Consider $f - \lambda . \mathrm{Id}_V \in \mathrm{Hom}_G(V, V)$. As $\mathrm{Ker}(f-\lambda.\mathrm{Id}_V)$ is a subrepn. of V and is non-zero ($v \in \mathrm{Ker}(f-\lambda.\mathrm{Id}_V)$) irreducibility of V implies that $V = \mathrm{Ker}(f-\lambda\,\mathrm{Id}_V)$

$\Rightarrow f = \lambda . \mathrm{Id}_V$.

$\square$

(16.5) 1-dim'l representations. Let G be a group.

$$\{\text{1-dim'l repns. of } G\} \longleftrightarrow \{\text{Group homs. } G \to GL_1(\mathbb{C}) = \mathbb{C}^\times\}$$

as $\mathbb{C}^\times$ is abelian, we further get

$$\{\text{1-dim'l repns. of } G\} \longleftrightarrow \{\text{Group homs. } G/(G,G) \to \mathbb{C}^\times\}$$

eg. if $G = \mathbb{Z}/n\mathbb{Z}$, a gp. hom $G \to \mathbb{C}^\times$ is determined by $z \in \mathbb{C}^\times$ such that $z^n = 1$.

$$\{\text{Gp. homs. } \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times\} = \{e^{\frac{2\pi i k}{n}} : 0 \leq k \leq n-1\}$$