Lecture 17

(17.0) Summary of results so far:

- $G \circlearrowright V$ is same as having a group hom $\rho : G \to GL(V)$
  - V (or (ρ, V) to be more precise) is a representation of G.
- Natural operations on vector spaces extend to G-representations (section 15.3 p.4)
- G: finite and V: finite-dimensional ; $G \circlearrowright V$ implies $V = V_1 \oplus \cdots \oplus V_\ell$ for some irreducible G-subrepns $V_1, \ldots, V_\ell$ of V.

[we obtained this as a consequence of Maschke's Thm - Theorem 16.2 p3]

- Schur's Lemma. (Lemma 16.4 p.5)

$f \in \mathrm{Hom}_G(V, W)$ $\Rightarrow$ $f = 0$ or f is an iso.

$G \circlearrowright V, W$ irreducible repns.

If additionally V (and W) are finite-dim'l, $f \in \mathrm{End}_G(V)$, then $\exists\, \lambda \in \mathbb{C}$ such that $f = \lambda \cdot \mathrm{Id}_V$

(17.1) Cor. of Schur's Lemma: Let V be a f.d. repn. of a finite group G. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be the set of (iso. classes) of irreducible finite-dimensional representations of G. Then $V = \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus m_\lambda}$ where

$$m_\lambda = \dim \operatorname{Hom}_G(V, V_\lambda) \quad (= \dim \operatorname{Hom}_G(V_\lambda, V))$$

Proof. Since every finite-dimensional representation of G is iso. to a direct sum of irreducible f.d. repns; there are non-negative integers $\{m_\lambda\}_{\lambda \in \Lambda}$ such that $V \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus m_\lambda}$.

Now $\operatorname{Hom}_G(V, V_\lambda) \simeq \bigoplus_{\lambda' \in \Lambda} \operatorname{Hom}_G(V_{\lambda'}, V_\lambda)^{\oplus m_{\lambda'}}$ [Check this!]

Now $\operatorname{Hom}_G(V_{\lambda'}, V_\lambda) = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ \mathbb{C} & \text{if } \lambda = \lambda' \end{cases}$

$\Rightarrow \operatorname{Hom}_G(V, V_\lambda) \simeq \operatorname{Hom}_G(V_\lambda, V_\lambda)^{\oplus m_\lambda} \simeq \mathbb{C}^{m_\lambda}$

and the claim follows $\square$

For later reference, let us record this

- $\Lambda :=$ the set of iso. classes of irr. f.d. repns. of G (finite gp.)
  $\lambda \in \Lambda \leftrightarrow V_\lambda \circlearrowleft G \quad (\rho_\lambda : G \longrightarrow GL(V_\lambda))$
- $\dim \operatorname{Hom}_G(V_\lambda, V_\mu) = \delta_{\lambda\mu}$

(17.2) Regular representation.

Let G be a group, consider $G \circlearrowright G$ by left multiplication. We promote this action to a repn. of G as usual (see the example of §15.1 p.1)

$\mathbb{C}G := \mathrm{Fun}(G) =$ function, $\mathbb{C}$-valued, on set G.

$$G \circlearrowright \mathbb{C}G : \quad (g \cdot f)(x) = f(g^{-1}x) \quad \forall \begin{smallmatrix} g, x \in G \\ f \in \mathrm{Fun}(G) \end{smallmatrix}$$

– called the regular (or left regular) representation.

Assume $|G| < \infty$. Consider function $e_g \in \mathrm{Fun}(X) = \mathbb{C}G$

$$e_g(x) = \delta_{g,x} = \begin{cases} 1, & \text{if } g = x \\ 0, & \text{o/w} \end{cases}$$

$\{e_g\}_{g \in G}$ is a basis of $\mathbb{C}G$. $\forall\ \sigma \in G$

$$(\sigma \cdot e_g)(x) = e_g(\sigma^{-1}x) = \delta_{g, \sigma^{-1}x} = \delta_{\sigma g, x}$$
$$= e_{\sigma g}$$

finite gp.

Prop. $\forall\ G \circlearrowright V$ (for every f.d. repn. V of G)

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathbb{C}G, V) & \simeq & V \\ \Psi & & \Psi \\ X & \longmapsto & X(e_1) \end{array}$$

$1 \in G$ the identity elt.

Proof. The map
$$\begin{array}{ccc} \mathrm{Hom}_G(\mathbb{C}G, V) & \longrightarrow & V \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \longmapsto & X(e_1) \end{array}$$
is clearly a linear map. It is injective, since $X(e_1) = 0$

$$\Rightarrow \quad X(e_g) = X(g \cdot e_1) = g\,X(e_1) = 0 \quad \forall g \in G.$$

As $|G| < \infty$ and hence $\{e_g\}_{g\in G}$ is a basis of $\mathbb{C}G$,

$X(e_g) = 0 \ \forall g \in G \Rightarrow X = 0.$

Now we prove that this map is surjective. Let $v \in V$.

Define $X_v : \mathbb{C}G \longrightarrow V$ by $X_v(e_g) = g \cdot v$ on the basis $\{e_g\}_{g\in G}$. Now $\forall \sigma \in G$,

$$\forall g \in G: \ (\sigma \cdot X_v)(e_g) = \sigma\left(X_v(\sigma^{-1} \cdot e_g)\right) = \sigma\left(X_v(e_{\sigma^{-1}g})\right)$$
$$= \sigma(\sigma^{-1}g) \cdot v = g \cdot v = X_v(e_g)$$

$\Rightarrow X_v$ is a G-intertwiner.

Moreover $X_v \longmapsto X_v(e_1) = v \in V$ proves surjectivity.

$\square$

(17.3) Combining Prop. (17.2) and Cor (17.1) we get

Theorem [Decomposition of regular representations]

G : finite group. Then:

$$\mathbb{C}G \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus d_\lambda} \quad \text{where}$$

$d_\lambda = \dim(V_\lambda) \quad \forall \lambda \in \Lambda.$

Proof. Theorem 16.2 p.3 $\Rightarrow \quad \mathbb{C}G = \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus m_\lambda}$

By Cor (17.1) $\quad m_\lambda = \dim \mathrm{Hom}_G(\mathbb{C}G, V_\lambda)$

(by Prop. 17.2 $\quad = \dim V_\lambda = d_\lambda$ $\square$

Cor. $\quad \sum_{\lambda \in \Lambda} (\dim V_\lambda)^2 = |G|$

in particular, Λ is finite.

(17.4) Explicit map: $\forall \lambda \in \Lambda$, define

$$\varphi_\lambda : V_\lambda \otimes V_\lambda^* \longrightarrow \mathbb{C}G$$

$$(v \otimes \xi) \longmapsto \{g \mapsto \xi(g.v)\} \in \mathrm{Fun}(G) \overset{\mathbb{C}G}{=}$$

[called matrix coefficients]

Prop. $\varphi : \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^* \xrightarrow[\sum_{\lambda\in\Lambda}\varphi_\lambda]{\sim} \mathbb{C}G$

Proof. $\rho_\lambda : G \longrightarrow GL(V_\lambda)$ is an irr. fd repn. of G ($\forall \lambda\in\Lambda$)

Consider
$$\mathbb{C}G \xrightarrow{\psi} \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda)$$
$$\sum_{g\in G} f_g \cdot e_g \longmapsto \Big(\sum_{g\in G} f_g\, \rho_\lambda(g)\Big)_{\lambda\in\Lambda}$$

Claim : ψ is injective.

Proof of the claim : Let $f = \sum_{g\in G} f_g e_g \in \mathbb{C}G$ be such that $\psi(f) = 0$, i.e. $\sum_{g\in G} f_g \rho_\lambda(g) = 0 \quad \forall \lambda\in\Lambda$.

Since every f.d. repn. of G is iso. to a direct sum of irr. fd. repns., $\sum_{g\in G} f_g\, \rho(g) = 0 \quad \forall\ \rho : G \to GL(V)$ a f.d. repn

Take (ρ, V) to be the regular repn. $\mathbb{C}G$, and act on e_1 to get $\sum_{g\in G} f_g e_g = 0$ as required. The claim is proved.

Combining with Cor (17.3) p.5, we get that ψ is an iso.

Now $\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda) \simeq \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^*$ by Hom-Tensor adjointness. The reader should verify that composing ψ^{-1} with the natural iso $V_\lambda \otimes V_\lambda^* \simeq \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda)$ gives us φ. This completes the proof of the prop. $\square$

(17.5) Keeping track of the group action

- $\mathbb{C}G \simeq \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^*$ as G-repns. when G acts on $V_\lambda \otimes V_\lambda^*$ as: $g\cdot(v\otimes\xi) = (g\cdot v)\otimes \xi$.

- Let G act on $\mathbb{C}G$ via conjugation: i.e.

$$\underset{\text{conjugation}}{G \circlearrowright G} \rightsquigarrow G \circlearrowright \mathrm{Fun}(G) = \mathbb{C}G$$

$C(g)(x) = gxg^{-1}$
$(\forall\, g, x \in G)$

$$[C(g)\cdot f](x) = f(C(g^{-1})(x)) = f(g^{-1}xg) \qquad (\forall\, f\in\mathbb{C}G;\ g,x\in G)$$

Then $\mathbb{C}G_{conj} \simeq \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^*$ as G-repns. where G acts on $V_\lambda \otimes V_\lambda^*$ as usual $g\cdot(v\otimes\xi) = (g\cdot v)\otimes(g\cdot\xi)$

Proof $\varphi: \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^* \xrightarrow{\sim} \mathbb{C}G$ is given by

$$\forall v \in V_\lambda,\ \xi \in V_\lambda^*, \qquad \varphi_\lambda(v\otimes\xi) : G \longrightarrow \mathbb{C} \quad (\in \mathbb{C}G)$$
$$g \longmapsto \xi(g\cdot v)$$

Let $\sigma \in G$. Then $\forall\, g \in G$

$$\varphi_\lambda\left(\overset{\sigma}{\cancel{g}}\cdot(v\otimes\xi)\right)(g) = \varphi_\lambda(\sigma\cdot v \otimes \sigma\cdot\xi)(g)$$
$$= (\sigma\cdot\xi)(g\sigma\cdot v) = \xi(\sigma^{-1}g\sigma\cdot v)$$
$$= \varphi_\lambda(v\otimes\xi)(\sigma^{-1}g\sigma) = \left[C(\sigma)\cdot\varphi_\lambda(v\otimes\xi)\right](g)$$

$\Rightarrow \varphi_\lambda$ is an intertwiner. $\square$