Lecture 18

(18.0) Recall: last time we proved that

$$(*)\quad \mathbb{C}G \cong \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^* \quad \text{where } G \text{ is a finite}$$

group; $\mathbb{C}G$ is a vector space (f.d.) of $\mathbb{C}$-valued functions on G and $\{V_\lambda\}_{\lambda\in\Lambda}$ are (all) mutually non-isomorphic f.d. irred. repns. of G.

There are two main G-actions on $\mathbb{C}G$:

(i) $G \circlearrowright G$ by left mult. $\rightsquigarrow$ $G \circlearrowright \mathbb{C}G$ (left) regular representation. Then the iso. $(*)$ is a G-intertwiner if

$$G \circlearrowright \underset{\substack{\uparrow\\ \text{act here}}}{V_\lambda} \otimes \underset{\substack{\uparrow\\ \text{leave it alone}}}{V_\lambda^*} \quad \text{by} \quad g.(v\otimes\xi) = (g.v)\otimes\xi$$

(ii) $G \circlearrowright G$ by conjugation. $\rightsquigarrow$ $G \circlearrowright \mathbb{C}G_{(\text{conj.})}$

$(*)$ is then a G-intertwiner if $G \circlearrowright V_\lambda \otimes V_\lambda^*$ as usual $\big(g.(v\otimes\xi) = (g.v)\otimes(g.\xi)\big)$.

(18.1) Class functions are the fixed points of $G \circlearrowright \mathbb{C}G_{(\text{conj.})}$

i.e. $f \in \mathbb{C}G$ is a <u>class fn.</u> if $f(x) = f(gxg^{-1})\ \forall g,x\in G$.

$\mathbb{C}G_{\text{class}} \subset \mathbb{C}G$ subspace of class functions

Proposition – $\mathbb{C}G_{\text{class}} \simeq \bigoplus_{\lambda\in\Lambda} \mathbb{C}$

In particular, $|\Lambda| = \dim \mathbb{C}G_{\text{class}} =$

$= \#$ of conjugacy classes in G.

Proof.

$$\begin{array}{cccccc}
 & \mathbb{C}G_{(\text{conj})} & \simeq & \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^* & \simeq & \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda) \\
 & \cup & & \cup & & \cup \\
(\mathbb{C}G)^{G\text{-conj.}} = & \mathbb{C}G_{\text{class}} & \simeq & \bigoplus_{\lambda\in\Lambda} (V_\lambda \otimes V_\lambda^*)^G & \simeq & \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda)^G
\end{array}$$

[Problem 4 of Set 6]

By Schur's lemma (Lemma 16.4 p.5) $\mathrm{Hom}_G(V_\lambda, V_\lambda) \simeq \mathbb{C}\cdot \mathrm{Id}_{V_\lambda}$

(recall: problem 5 of Set 6: $\mathrm{Hom}_{\mathbb{C}}(V,W)^G = \mathrm{Hom}_G(V,W)$)

Hence the proposition follows. □

e.g. Recall that # of conjugacy classes in S_n

$= \#$ of partitions of n $(=: p(n))$

Thus S_n as $p(n)$ mutually non-iso irred. f.d. repns

(e.g. $n=5$, $p(5) = \#\{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}$

$= 7$)

e.g. G finite abelian $\Rightarrow$ Conj. classes $= \{\{g\} : g \in G\}$

$\Rightarrow$ # of irr f.d. repns $= |G|$

Since $\sum_{\lambda \in \Lambda_G} (\dim V_\lambda)^2 = |G|$ and $|\Lambda_G| = |G|$

we get that $\dim V_\lambda = 1 \quad \forall \lambda \in \Lambda_G$.

e.g. $G = S_3$. $|\Lambda_G| = |\{(3), (2,1), (1,1,1)\}| = 3$

We already know 2 irred. f.d. repns. Trivial repn: $\begin{matrix} G \to \mathbb{C}^\times \\ \sigma \mapsto 1 \end{matrix}$ $(\forall \sigma \in G)$

& Sign repn. $\text{sign} : \begin{matrix} G \longrightarrow \mathbb{C}^\times \\ \sigma \longmapsto \text{sign}(\sigma) \end{matrix}$. Let V be the remaining irred. f.d. repn. Then $6 = |G| = 1 + 1 + (\dim V)^2$

$\Rightarrow \dim V = 2$.

- $V = \{a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 : a_1 + a_2 + a_3 = 0\} \subset \mathbb{C}^3 \circlearrowright S_3$

$\uparrow$ 2-dim'l irreducible repn. of S_3

[permute basis vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$]

(18.2) Character of a repn

Let $\rho : G \to GL(V)$ be a finite-dim'l repn. of a group G.

Definition: The character of V, denoted by χ_V, is a class function on G given by

$$\chi_V(g) = \text{Trace}(\rho(g))$$

$$\underset{\text{f.d. repn}}{G \circlearrowright V} \rightsquigarrow \chi_V \in \mathbb{C}G_{\text{class}}$$

[Proof that χ_V is a class function: $\chi_V(gxg^{-1})$ $= \text{Tr}(\rho(gxg^{-1})) = \text{Tr}(\rho(g)\rho(x)\rho(g)^{-1}) = \text{Tr}(\rho(x))$ $= \chi_V(x) \quad \forall g, x \in G$]

Prop. [Basic properties of characters]

(i) $\chi_V(1) = \dim V$ ($1 \in G$ is the identity elt.)

(ii) $\chi_{V \oplus W} = \chi_V + \chi_W$

(iii) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

(iv) $\chi_{V^*}(g) = \chi_V(g^{-1}) \quad \forall g \in G$

[Proof of (iv), the result follow easily from the similar properties of Trace,: $\chi_{V^*}(g) = \text{Trace of } g|_{V^*}$

recall : $G \circlearrowright V^*$ by $(g\cdot\xi)(v) = \xi(g^{-1}\cdot v)$

$(\forall\, g\in G,\ \xi\in V^*,\ v\in V)$

for a matrix $X \in End(V)$, $Trace(X) := \sum_{i=1}^{m} \xi_i\,(X v_i)$ where $\{v_1, \dots, v_m\}$ is a basis of V and $\{\xi_i = v_i^*\}_{1\le i\le m}$ is the dual basis (of V^*)

$$\text{Thus}\quad Trace(g \text{ acting on } V^*) = \sum_{i=1}^{m} (g\cdot\xi_i)(v_i)$$

(here we identify V^{**} with V and $v_i \leftrightarrow \xi_i^*$; as V is finite-dim'l)

$$= \sum_{i=1}^{m} \xi_i(g^{-1} v_i)$$

$$= Trace(g^{-1} \text{ acting on } V)$$

□

(18.3) Average of characters

Proposition. Let V be a finite-dimensional repn. of a finite group G. Then

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g) = \dim V^G$$

Proof. Average :

$$\begin{tikzcd} V \arrow[r] & V \\ v \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \frac{1}{|G|}\sum_{g\in G} g.v \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Check :
- Average is a G- intertwiner
- Image of Average $= V^G$
- Average $(v) = v \quad \forall\, v \in V^G$

i.e. if we write $V = V^G \oplus W$ as a vector space then

$$\text{Average} = \begin{array}{c|c} \left[\begin{array}{c|c} \text{Id} & * \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} V^G \\ W \end{array} \\ \begin{array}{cc} V^G & W \end{array} & \end{array} \Rightarrow \text{Trace}(\text{Average}) = \dim V^G .$$

But
$$\text{Trace}(\text{Average}) = \frac{1}{|G|} \sum_{g \in G} \text{Trace}(g \text{ acting on } V)$$
$$= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \qquad \square$$

(18.4) ~~Prop.~~ Definition : We introduce a symmetric, bilinear form $(\cdot\,,\cdot) : \mathbb{C}G \times \mathbb{C}G \longrightarrow \mathbb{C}$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1})\, f_2(g)$$

Prop. Let V, W be two f.d. repns of a finite group G. Then

$$(\chi_V, \chi_W) = \dim \operatorname{Hom}_G (V, W)$$

Proof $(\chi_V, \chi_W) = \frac{1}{|G|} \sum_{g\in G} \chi_V(g^{-1})\, \chi_W(g)$

$$\underset{\text{(by Prop. (18.2) p.4)}}{=} \frac{1}{|G|} \sum_{g\in G} \chi_{V^*\otimes W}(g)$$

$$\underset{\text{(by Prop (18.3) p.5)}}{=} \dim (V^* \otimes W)^G \underset{\substack{\text{(Problems 4 \& 5} \\ \text{of Set 6)}}}{=} \dim \mathrm{Hom}_G(V, W)$$

□

Cor. (of Prop. 18.4 and Schur's Lemma - see p.2 of Lecture 17)

(1) $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis of $\mathbb{C}G_{\text{class}}$.

(here $\chi_\lambda = \chi_{V_\lambda}$)

(2) Let V be a finite-dim'l repn. of G. Then V is irreducible if, and only if $(\chi_V, \chi_V) = 1$.

[Proof of (2): $V \simeq \bigoplus_{\lambda\in\Lambda} V_\lambda^{\oplus m_\lambda}$ by Thm. 16.2 p.3

$$(\chi_V, \chi_V) = \sum_{\lambda\in\Lambda} m_\lambda^2 = 1 \iff \exists\, \mu \in \Lambda \text{ st. } m_\mu = 1 \;\&\; m_\lambda = 0 \;\forall \lambda \neq \mu.$$

$\iff V \simeq V_\mu$ is irreducible]