Lecture 19

(19.0) Recall the following definitions

$$\mathbb{C}G = \{ f : G \to \mathbb{C} \}$$

$$\cup$$

$$\mathbb{C}G_{class} = \{ f : G \to \mathbb{C} \text{ such that } f(\sigma x \sigma^{-1}) = f(x) \ \forall \sigma, x \in G \}$$

For $G \circlearrowright V$ a f.d. repn., $\chi_V \in \mathbb{C}G_{class}$ is defined as

$$\chi_V(g) = \text{Trace of } g \text{ acting on } V.$$

Λ_G := the set of isomorphism classes of irreducible finite-dimensional repns of G.

$(\cdot,\cdot) : \mathbb{C}G \times \mathbb{C}G \longrightarrow \mathbb{C}$ is symmetric, bilinear form:

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g)$$

Summary of results about characters: (Prop. 18.2, 18.3, 18.4)

(i) $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ (ii) $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

(iii) $\chi_{V^*}(g) = \chi_V(g^{-1})$

(iv) $\displaystyle \frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \dim V^G$

(v) $(\chi_{V_1}, \chi_{V_2}) = \dim \mathrm{Hom}_G(V_1, V_2)$

(vi) $\{\chi_\lambda := \chi_{V_\lambda}\}_{\lambda \in \Lambda_G}$ is an o.n. basis of $\mathbb{C}G_{\text{class}}$

(19.1) Cor. Let V be a f.d. repn. of G. Then

$$(\chi_V, \chi_V) = \sum_{\lambda \in \Lambda_G} (\text{Multiplicity of } V_\lambda \text{ in } V)^2$$

$$\left(\text{i.e. } \sum_{\lambda \in \Lambda_G} m_\lambda^2 \quad \text{if } V \simeq \bigoplus_{\lambda \in \Lambda_G} V_\lambda^{\oplus m_\lambda} \right)$$

In particular V is irreducible iff $(\chi_V, \chi_V) = 1$.

(19.2) Character table of G is a square - $\ell \times \ell$ table of numbers, where $\ell = |\Lambda_G| = |\text{Conj. classes in } G|$

$\Lambda_G \;\#\; \{V_1, \ldots, V_\ell\}$ irr. f.d. repns (mutually non-iso.)

$\{C_1, \ldots, C_\ell\}$ conjugacy classes in G

$(i,j)^{\text{th}}$ entry of character table $= \chi_{V_j}(C_i)$

(recall: it is constant on conjugacy classes)

(19.0) (vi) $\Rightarrow$ Columns of character table are orthonormal.

(19.3) Example. $G = S_4$

Conjugacy class $= \{(1,1,1,1),\ (2,1,1),\ (2,2),\ (3,1),\ (4)\}$
cycle-type

$\#$ of elements : $(1,1,1,1) \leftarrow$ has 1 element $(1 \in S_4)$

$(2,1,1)$: has $\binom{4}{2} = 6$ elts. $(2,2)$: has $\frac{1}{2}\binom{4}{2} = 3$ elts

$(3,1)$: has $4 \cdot 2 = 8$ elts (4) : has $3\cdot 2 \cdot 1 = 6$ elts

We already know two irred. repns. of S_4 : trivial & sign

Triv: $S_4 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$, $\sigma \longmapsto 1 \ (\forall \sigma \in S_4)$

Sgn: $S_4 \longrightarrow \mathbb{C}^\times$, $\sigma \longmapsto \mathrm{sign}(\sigma)$

$S_4 \circlearrowright \mathbb{C}^4$ by permuting the basis vectors, i.e.

$$S_4 \longrightarrow GL_4(\mathbb{C})$$
$$\sigma \longmapsto X_\sigma \quad (i,j)^{th} \text{ entry} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{o/w} \end{cases}$$

$\Rightarrow$ Trace of σ acting on $\mathbb{C}^4 = \sum_{i=1}^{4} (X_\sigma)_{ii}$

$= \#\{j : \sigma(j) = j\} = |\{1,\dots,4\}^\sigma|$

Conj. class	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
$\chi_{\mathbb{C}^4}$	4	2	0	1	0

$$(\chi_{\mathbb{C}^4}, \chi_{\mathbb{C}^4}) = \frac{1}{24}\left\{1\cdot(4)^2 + 6\cdot(2)^2 + 3\cdot(0)^2 + 8\cdot(1)^2 + 6\cdot(0)^2\right\}$$

\# of elts in conj. class (1,1,1,1)

$\chi_{\mathbb{C}^4}(1)\cdot\chi_{\mathbb{C}^4}(1)$

$$= \frac{1}{24}\left(16+24+8\right) = 2$$

$\Rightarrow$ $\mathbb{C}^4$ has exactly two (mutually non-iso.) irreducible summands. One of them is the trivial repn $\left(= \left\{\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} : a \in \mathbb{C}\right\}\right)$

Let V_1 be the other (3-dim'l irred.)

$\chi_{V_1} = \chi_{\mathbb{C}^4} - \chi_{triv}$:

(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
3	1	−1	0	−1

We can take $V_2 = V_1 \otimes \mathrm{sgn}$ $\leftarrow$ again 3-d. irr. (as sgn is 1-dim'l)

	# elts	χ_{triv}	χ_{sgn}	χ_{V_1}	χ_{V_2}	V_3
(1,1,1,1)	1	1	1	3	3	2
(2,1,1)	6	1	−1	1	−1	
(2,2)	3	1	1	−1	−1	
(3,1)	8	1	1	0	0	
(4)	6	1	−1	−1	1	

(Trace of Id = dim)

$\chi_{V_3}(1)$

$\|$

To complete the last column we used

$$1^2 + 1^2 + 3^2 + 3^2 + (\dim V_3)^2 = |S_4| = 24 \Rightarrow \dim V_3 = 2$$

and the fact that the last column is $\perp$ to other 4.

	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ_{V_3}	2	0	2	−1	0

(19.4) Orthogonality of rows.

Let $g_1 \cdot g_2 \in G$ and let us compute

$$\cancel{\frac{1}{|G|}} \sum_{\lambda \in \Lambda_G} \chi_{V_\lambda}(g_1^{-1})\, \chi_{V_\lambda}(g_2) = \frac{1}{|G|} \sum_{\lambda \in \Lambda_G} \text{Trace of } g_1 \otimes g_2 \text{ acting on } V_\lambda^* \otimes V_\lambda$$

$$= \cancel{\frac{1}{|G|}} \text{ Trace of } g_1 \otimes g_2 \text{ acting on } \bigoplus_{\lambda \in \Lambda_G} V_\lambda^* \otimes V_\lambda$$

By problem 1 of Set 7, part (2)

$$\bigoplus_{\lambda \in \Lambda_G} V_\lambda^* \otimes V_\lambda \simeq \mathbb{C}G = \text{Functions } G \to \mathbb{C}$$

$$\circlearrowright$$

$$[(g_1, g_2) \cdot f](\sigma) = f(g_1^{-1} \sigma g_2)$$

Let $\{e_g\}_{g \in G}$ be the usual basis of $\mathbb{C}G$ $\left(e_g(x) = \delta_{g,x} = \begin{cases} 1 & \text{if } g = x \\ 0 & \text{o/w} \end{cases}\right)$

Then $(g_1, g_2) \cdot e_g = e_{g_1 g g_2^{-1}}$

$\Rightarrow$ Trace of (g_1, g_2) acting on $\mathbb{C}G$

$= \left| \{ x \in G \; : \; g_1 x g_2^{-1} = x \} \right| \; = 0$ if g_1 and g_2 are not conjugate to each other.

$= \left|$ Centralizer of (any) g from conjugacy class $C \right|$

if $g_1, g_2 \in C$. Thus we have proved

Prop.
$$\sum_{\lambda \in \Lambda_G} \chi_{V_\lambda}(g_1^{-1}) \, \chi_{V_\lambda}(g_2) = \begin{cases} 0 & \text{if } g_1, g_2 \text{ are from different conj. classes} \\ |Z_G(g)| & \text{if } g_1, g_2 \in C \\ & (g \in C \text{ any}) \end{cases}$$

($Z_G(g)$ = Centralizer of g in G
$= \{ x \in G \; : \; gx = xg \}$)