Lecture 20

(20.0) Recall: we defined the character table of a finite group G and proved orthonormality of its columns and rows. (see Prop (19.4) page 6).

(20.1) Consider the representation of $S_n \circlearrowright \mathbb{C}^n$.

Recall that we proved (see example (15.3) page 5).

$$\mathbb{C}^n \simeq \underset{\substack{\uparrow \\ \text{trivial repn}}}{\mathbb{C}} \oplus \underset{\substack{\uparrow \\ n-1 \text{ dim'l repn.}}}{V}$$

We claim that V is irreducible. For this it suffices to prove that $(\chi_{\mathbb{C}^n}, \chi_{\mathbb{C}^n}) = 2$ (see the argument on page 4 of Lecture 19)

Now $\chi_{\mathbb{C}^n}(\sigma) = |X^\sigma|$ $\left\{ \begin{array}{l} X = \{1, \ldots, n\} \circlearrowleft S_n \\ X^\sigma = \{x \in X : \sigma(x) = x\} \end{array} \right.$

(p.3 Lecture 19)

So $$(\chi_{\mathbb{C}^n}, \chi_{\mathbb{C}^n}) = \frac{1}{n!} \sum_{\sigma \in S_n} |X^\sigma|^2$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} |(X \times X)^\sigma|$$

(σ acts on $X \times X$ by $\sigma \cdot (x,y) = (\sigma x, \sigma y)$)

$$= \left| S_n \backslash X \times X \right| \quad \text{by Burnside's counting lemma}$$
(Lemma (2.6) page 7)

$$= \Big| \underbrace{\{(x,x) : x \in X\} \sqcup \{(x,y) : x \neq y \in X\}}_{2\text{- orbits}} \Big| = 2$$

as required.

(20.2) Restriction and Induction of representations.

Let $\varphi : G_1 \longrightarrow G_2$ be a group homomorphism and let $\rho : G_2 \longrightarrow GL(V)$ be a representation.

The <u>pull-back representation</u> of G_1, denoted by $\varphi^*(V)$, is obtained by composing φ & ρ

$$G_1 \xrightarrow{\rho \circ \varphi} GL(V)$$

In particular, if $G_1 = H \hookrightarrow G = G_2$ is a subgroup we get <u>the restriction</u>, $\mathrm{Res}_H^G (V)$:

$$\rho : G \to GL(V) \quad \rightsquigarrow \quad \mathrm{Res}_H^G(\rho) : H \longrightarrow GL(V)$$
(ρ restricted to H).

Conversely, let $H < G$ be a subgroup and let $\pi : H \longrightarrow GL(W)$ be a representation.

The <u>induced representation</u>,

$$\mathrm{Ind}_H^G(\pi) : G \longrightarrow GL(\mathrm{Ind}_H^G(W)) \text{ is}$$

defined as:

- $\mathrm{Ind}_H^G(W) = \{ f : G \to W$ such that $f(gh) = \pi(h)^{-1} f(g) \; \forall g \in G, h \in H \}$

- $G \circlearrowright$ $\quad [\mathrm{Ind}_H^G(\pi)(\sigma)(f)](g) = f(\sigma^{-1} g)$

$\forall \, g, \sigma \in G; \; f : G \to W \in \mathrm{Ind}_H^G(W)$

(20.3) Several remarks.

- In restriction, the underlying vector space does not change, while in <u>inducing representations</u> it does.

- As we will see below, if W is finite-dimensional and $(G:H) < \infty$, we have

$$\dim \mathrm{Ind}_H^G (W) = \dim(W) \cdot (G:H)$$

- Careful: Res and Ind are <u>NOT</u> inverse to each other. ($\mathrm{Res}_H^G(\mathrm{Ind}_H^G(W)) \neq W$ even as a vector space) because of the jump in dimension.

(20.4) Lemma. Let $G = \bigsqcup_{i=1}^{\ell} g_i H$ $\quad (\ell = (G:H) < \infty)$

Then we have an iso (<u>NOT</u> natural - it depends on the choice of coset representatives $g_1, \dots, g_\ell$) of vector spaces (<u>NOT</u> representations).

$$\begin{array}{ccc} \mathrm{Ind}_H^G (W) & \simeq & \overbrace{W_{(1)} \oplus \dots \oplus W_{(\ell)}}^{\ell\text{-terms}} \quad \left[\begin{array}{c}\text{each} \\ W_{(j)} \cong W\end{array}\right] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & (f(g_1), \dots, f(g_\ell)) \end{array}$$

$$\left[W_{(j)} = \Big\{ \underset{\in \mathrm{Ind}_H^G(W)}{f : G \to W} \text{ such that } \begin{array}{l} f(g_i) = 0 \\ \forall i \neq j \end{array} \Big\} \right]$$

The proof is left as an exercise.

(20.5) Example.

$$G = D_5 = \left\{ 1, \begin{matrix} s_1, & s_1 s_2, & s_1 s_2 s_1, & s_1 s_2 s_1 s_2 \\ s_2, & s_2 s_1, & s_2 s_1 s_2, & s_2 s_1 s_2 s_1 \end{matrix} \quad \begin{matrix} s_1 s_2 s_1 s_2 s_1 \\ = s_2 s_1 s_2 s_1 s_2 \end{matrix} \right\}$$

$$\vee$$

$$H = \left\{ 1, \begin{matrix} s_1 s_2 & , s_1 s_2 s_1 s_2 \\ s_2 s_1 & , s_2 s_1 s_2 s_1 \end{matrix} \right\} \simeq \mathbb{Z}/_5\mathbb{Z}$$

$$s_1 s_2 \longmapsto \bar{1}$$

$$G/H = 1 \cdot H \sqcup s_1 \cdot H \quad \text{(choice of coset representatives)}$$

Let $\xi : H \longrightarrow GL_1(\overset{W}{\mathbb{C}}) = \mathbb{C}^\times$ be 1-dim'l irr repn. of H

$$s_1 s_2 \longmapsto \zeta \in \{1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}\}$$

$\rightsquigarrow$ 2-dim'l repn. of G $= \mathrm{Ind}_H^G(\xi)$.

$$V = \mathrm{Ind}_H^G(W) = \{ f : G \to \mathbb{C} \text{ such that } f(gh) = \xi(h)^{-1} f(g) \;\; \forall g \in G,\, h \in H \}$$

$$V \simeq \mathbb{C} \oplus \mathbb{C}$$

$$f \longmapsto (f(1), f(s_1))$$

That is, we have chosen a basis of V

$f_1 \in V$ defined by $f_1(1) = 1$, $f_1(s_1) = 0$

$f_2 \in V$ " " $f_2(1) = 0$, $f_2(s_1) = 1$

Let us compute the action of $s_2 \in G$ on V in this basis.

$$(s_2 \cdot f)(x) = f(s_2^{-1} x) = f(s_2 x)$$

$\forall \; f : G \to \mathbb{C} \in V$

$x \in G$.

→ $\underline{s_2 \cdot f_1}$: $(s_2 \cdot f_1)(\sigma) = f_1(s_2 \sigma) = 0$ if $\sigma \in H$

if $\sigma = s_1 h \in s_1 H$, we get $f_1(s_2 s_1 h) = \xi(s_1 s_2) = \zeta$

$$\Rightarrow \quad s_2 \cdot f_1 = \zeta f_2$$

Similarly, $s_2 \cdot f_2 = \zeta^{-1} f_1$. The action of s_1 is easier to work out, as it was the coset rep. we picked.

$$\Rightarrow \quad G \longrightarrow GL_2(\mathbb{C})$$

$$s_1 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$s_2 \longmapsto \begin{bmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{bmatrix}$$