Lecture 21

(21.0) Recall that last time we introduced the notions of restriction and induction of representations.

Let G be a group and $H < G$ a subgroup. We assume that $|G| < \infty$.

Given V, a repn. of G $\rightsquigarrow$ $\mathrm{Res}_H^G V$, a repn. of H

$[\rho : G \longrightarrow GL(V)]$ $\qquad$ $[\rho|_H : H \longrightarrow GL(V)]$

More generally, if $\varphi : G_1 \longrightarrow G_2$ is a group hom, and $\rho : G_2 \to GL(V)$ a repn. of G_2, we get a repn. $\varphi^*(V)$ of G_1 by $G_1 \xrightarrow{\rho\circ\varphi} GL(V)$. $\mathrm{Res}_H^G(V) = i^*(V)$ when $i : H \hookrightarrow G$.

Conversely let $\pi : H \longrightarrow GL(W)$ be a (f.d.) repn. of H.

$$\mathrm{Ind}_H^G(W) := \left\{ f : G \longrightarrow W \text{ s.t. } \begin{array}{l} f(gh) = \pi(h)^{-1}(f(g)) \\ \forall\, g \in G,\ h \in H \end{array} \right\}$$

$G \circlearrowright$ via $\mathrm{Ind}_H^G \pi$

$$\left[\left(\mathrm{Ind}_H^G \pi\right)(g) \cdot f\right](x) = f(g^{-1}x)$$

(21.1) Our first theorem computes character of induced repn. denoted below by $\mathrm{Ind}_H^G \chi_\pi$

Theorem. $\forall\, g \in G$

$$\mathrm{Ind}_H^G \chi_\pi \;(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x^{+1} \in H}} \chi_\pi (x^{-1} g x^{+1})$$

Proof. Let us choose coset representatives of G/H $\{g_1 H, \dots, g_\ell H\}$. Recall that

$$\mathrm{Ind}_H^G W \xrightarrow{\sim} W_1 \oplus \dots \oplus W_\ell \quad \text{where}$$

$$W_j = \{ f \in \mathrm{Ind}_H^G W \; : \; f(g_k H) = 0 \;\; \forall\, k \neq j \}$$

I will also denote it by $W_{g_j \cdot H}$.

Also, $\begin{array}{ccc} W_j & \xrightarrow{\sim} & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & f(g_j) \end{array}$. First we note that

$$\mathrm{Ind}_H^G \pi \;(g) : W_{g_j H} \longrightarrow W_{g g_j H} \quad \text{since } \forall\, f \in W_{g_j H}$$

$$f(g^{-1} x) \neq 0 \iff g^{-1} x \in g_j H \iff x \in g g_j H$$

Hence $\mathrm{Ind}_H^G \pi(g)$ has diagonal blocks corresponding only to j $(1 \leq j \leq \ell)$ s.t. $g g_j H = g_j H$

i.e. $g_j^{-1} g g_j \in H$

$$\Rightarrow \mathrm{Tr}\left(\mathrm{Ind}_H^G \pi(g)\right) = \sum_{\substack{1 \leq j \leq \ell \\ \text{s.t. } g_j^{-1} g g_j \in H}} \mathrm{Tr}\left(\mathrm{Ind}_H^G \pi(g)\Big|_{W_j}\right)$$

Claim:

$$\begin{tikzcd} W_j \arrow[r, "\sim"] \arrow[d, "\mathrm{Ind}_H^G(\pi)(g)|"'] & W \arrow[d, "\pi(g_j^{-1} g g_j)"] \\ W_j \arrow[r, "\sim"] & W \end{tikzcd}$$

Thus $\mathrm{Tr}\left(\mathrm{Ind}_H^G \pi(g)\Big|_{W_j}\right) = \mathrm{Tr}\left(\pi(g_j^{-1} g g_j)\right)$

$$= \chi_\pi(g_j^{-1} g g_j)$$

If we want to write a formula without the choice of coset representatives, we will sum over all $x \in G$ s.t. $x^{-1} g x \in H$ and note that $\chi_\pi : H \to \mathbb{C}$ is constant on conjugacy classes, to get

$$\mathrm{Ind}_H^G \chi_\pi(g) = \frac{1}{|H|} \sum_{\substack{x \in G \text{ s.t.} \\ x^{-1} g x \in H}} \chi_\pi(x^{-1} g x)$$

Proof of the claim: $\forall\, f \in W_j$,

$$\mathrm{Ind}_H^G \pi(g) \cdot f \longmapsto f(g^{-1} g_j) \qquad \text{and}$$

$$\hat{W}_j \xrightarrow{\sim} \hat{W}$$

$$\pi(g_j^{-1} g g_j) \cdot f(g_j) = f(g_j\, g_j^{-1}\, g^{-1} g_j) = f(g^{-1} g_j) \qquad \square$$

Cor. (21.2) When $W =$ Trivial repn of H, say denoted by, Triv_H, then

$$\mathrm{Ind}_H^G \chi_{\mathrm{Triv}}(g) = \#\text{ fixed points of } g \text{ acting on } G/H$$

$$= |(G/H)^g| \quad \left(\text{in our usual notation } X^g = \{x \in X : gx = x\} \text{ for } G \circlearrowright X\right)$$

Proof: By Theorem (21.1)

$$\cancel{\chi^G_H} \quad \mathrm{Ind}_H^G \chi_{\mathrm{triv}}(g) = \sum_{j : g_j^{-1} g g_j \in H} 1 = |(G/H)^g|. \qquad \square$$

(21.3) Random Remarks. The result stated in Theorem (21.2) has several analogues in (algebraic) geometry. In principle, it follows the construction of vector bundles

$$H \circlearrowright W \quad \rightsquigarrow \quad \begin{array}{c} G \times_H W = G \times W / \sim : \\ \downarrow \\ G/H \end{array} \qquad \begin{array}{c} (g_1, w_1) \sim (g_2, w_2) \\ \Updownarrow \\ g_1^{-1} g_2 \in H \text{ and } \\ w_2 = g_1^{-1} g_2 (w_1) \end{array}$$

and states 'roughly' that information about an operator on (any) construction involving vector bundle $G \times_H W$ can be recovered from the same on the fixed points of the same operator.

(21.4) Relation between restriction and induction – Adjointeness → called Frobenius reciprocity:

Theorem. $\mathrm{Hom}_H(\mathrm{Res}^G_H V, W) \simeq \mathrm{Hom}_G(V, \mathrm{Ind}^G_H(W))$

for every $G \circlearrowright V$ a repn of G and $H \circlearrowright W$ a repn. of $H < G$ a subgroup.

Explicitly the map $(\rho : G \to GL(V)\ ;\ \pi : H \to GL(W))$

$$\mathrm{Hom}_H\left(\mathrm{Res}^G_H V, W\right) \xrightarrow{\ F\ } \mathrm{Hom}_G\left(V, \mathrm{Ind}^G_H W\right)$$

is given by:

$$A \longmapsto \tilde{A} : \tilde{A}(v) : G \to W$$

$$\tilde{A}(v)(g) = A(\rho(g)^{-1}.v)$$

Proof. F is well-defined, i.e. $\tilde{A}(v) \in \mathrm{Ind}^G_H(W)$ and $\tilde{A}$ is a G-intertwiner.

- $\tilde{A}(v)(gh) = A(\rho(h)^{-1}\rho(g)^{-1}v) \overset{A \text{ is } H\text{-intertwiner}}{=} \pi(h)^{-1}\left[\tilde{A}(v)(g)\right]$

  $\Rightarrow \tilde{A}(v) \in \mathrm{Ind}^G_H(W)$

- $\tilde{A}(\rho(g).v)$ as an element of $\mathrm{Ind}^G_H W$ maps

$$\sigma \in G \longmapsto A(\rho(\sigma)^{-1}\rho(g)v) \in W$$

and $\left(\mathrm{Ind}^G_H g\right) \cdot \tilde{A}(v)$ maps $\sigma \longmapsto \tilde{A}(v)(g^{-1}\sigma)$

$$= A(\rho(\sigma)^{-1}\rho(g)v) \in W \checkmark$$

so $\tilde{A}(\rho(g).v) = \left(\mathrm{Ind}^G_H \pi(g)\right)(\tilde{A}(v))$

$\Rightarrow \tilde{A} \in \mathrm{Hom}_G(V, \mathrm{Ind}^G_H W)$

Note that $A(v) = \tilde{A}(v)(1)$ ⟵ identity elt. of G

Thus $\tilde{A} \longmapsto \{ v \in V \longmapsto \tilde{A}(v)(1) \}$ is clearly an inverse of F. This proves that F is an iso $\square$

(21.5) Taking dimensions and using

$$(\chi_{V_1}, \chi_{V_2}) = \dim \mathrm{Hom}_G(V_1, V_2) \quad \left[\begin{matrix}\text{Prop. 18.4} \\ \text{page 6}\end{matrix}\right]$$

we get: <u>Cor</u>: $(\mathrm{Res}_H^G \chi_V, \chi_W) = (\chi_V, \mathrm{Ind}_H^G \chi_W)$

[read: restriction and induction are adjoint to each other]

(21.6) Cor. Multiplicity of an irreducible V_λ f.d. repn of G in Ind_H^G Trivial $= \dim V_\lambda^H$

Proof Mult. of V_λ in Ind_H^G Trivial

$$= \dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \mathrm{Triv}, V_\lambda) \quad \left[\begin{matrix}\text{Cor 17.1} \\ \text{page 2}\end{matrix}\right]$$

$$\left[= (\chi_{V_\lambda}, \mathrm{Ind}_H^G \chi_{triv}) \right] = \dim \mathrm{Hom}_H(\mathrm{Triv}, \mathrm{Res}_H^G(V_\lambda))$$

$$= \dim V_\lambda^H \qquad \square$$

e.g. recall we computed the character table of S_4.

S_4 has 5 irr. f.d. repns, of dim 1, 1, 3, 3, 2

Triv, Sign, $V_1 = \left\{ \begin{bmatrix} a \\ b \\ \cancel{-a-b} \\ c \\ -a-b-c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$, $V_2 = V_1 \otimes \text{sgn}$

and a 2-dim'l repn. V_3.

Let $V = \text{Ind}_{S_2 \times S_2}^{S_4} \mathbb{1}$ ← Trivial repn. is $6 = \dfrac{|S_4|}{|S_2 \times S_2|}$ dim'l

- $(\chi_{triv}, \chi_V) = \dim \text{Triv}^{S_2 \times S_2} = 1$
- $(\chi_{sgn}, \chi_V) = \dim (\mathbb{C})^{S_2 \times S_2}$ [each factor acts as mult. by -1]
  $= 0$
- $(\chi_{V_1}, \chi_V) = \dim V_1^{S_2 \times S_2} = \dim \left\{ \begin{bmatrix} a \\ a \\ -a \\ -a \end{bmatrix} : a \in \mathbb{C} \right\} = 1$
- $(\chi_{V_2}, \chi_V) = 0$ (by dim count $6 < 1+3+3$)

$\Rightarrow \quad V = V_3 \oplus V_1 \oplus \text{Trivial}$

"a concrete model for V_3" - upto "lower order terms".