Lecture 22

(22.0) Recall : we defined induction / restriction of repns:

for $H < G$ a subgroup

$$G \circlearrowright V \quad\rightsquigarrow\quad H \circlearrowright \mathrm{Res}_H^G(V)$$

$$H \circlearrowright W \quad\rightsquigarrow\quad G \circlearrowright \mathrm{Ind}_H^G(W)$$

• Character formula :

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x) \qquad \begin{bmatrix} \text{Thm. 21.1} \\ \text{page 2} \end{bmatrix}$$

If we choose coset representatives $G/H = \bigsqcup_{j=1}^{\ell} g_j H$, then

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \sum_{\substack{1 \leq j \leq \ell \\ g_j^{-1} g g_j \in H}} \chi_W(g_j^{-1} g g_j)$$

$g_j H \in G/H$ is fixed by $g \in G \circlearrowright G/H$

- Frobenius Reciprocity (Thm. 21.4 page 5)

$$\mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \simeq \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$$

for every $G \circlearrowright V$ and $H \circlearrowright W$. Taking dimensions, we get

$$(\chi_{\mathrm{Res}_H^G(V)}, \chi_W) = (\chi_V, \chi_{\mathrm{Ind}_H^G(W)})$$

L.H.S. = bilinear form on $\mathbb{C}H_{class}$

R.H.S. = bilinear form on $\mathbb{C}G_{class}$

That is,

$$\mathbb{C}H_{class} \underset{\mathrm{Res}_H^G}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} \mathbb{C}G_{class}$$

are adjoint to each other.

(22.1) Restriction to abelian subgroups - Problem 9 of Set 7

Let $H < G$ be an abelian subgroup and $G \circlearrowright V$ $[\rho : G \to GL(V)]$ be a finite-dimensional irreducible repr.

Choose an eigenvector for $H \circlearrowright V$. That is, since $\{\rho(h) \in GL(V)\}_{h \in H}$ is a family of commuting matrices they have a joint eigenvector $v \in V$ $(v \neq 0)$

$$\rho(h)(v) = \xi(h) \cdot v \quad \text{for a group hom. } \xi : H \to \mathbb{C}^\times$$

Define

$$\begin{array}{rccc} \phi : & \mathrm{Ind}_H^G(\xi) & \longrightarrow & V \\ & \Psi & & \Psi \\ & f & \longmapsto & \dfrac{1}{|G|} \displaystyle\sum_{g \in G} f(g) \cdot \rho(g)(v) \end{array}$$

(so $f : G \to \mathbb{C}$ satisfies
$f(gh) = \xi(h)^{-1} f(g) \ \forall \, g \in G, h \in H$)

Check: $\phi(\sigma \cdot f) = \rho(\sigma)(\phi(f)) \quad \forall \, \sigma \in G, f \in \mathrm{Ind}_H^G(\xi)$

$$\Big[\ \phi(\sigma \cdot f) = \frac{1}{|G|} \sum_{g \in G} f(\sigma^{-1} g)(\rho(g)(v))$$

$$= \frac{1}{|G|} \sum_{x \in G} f(x) \, \rho(\sigma x)(v) = \rho(\sigma)(\phi(f)) \ \Big]$$

Also, if $f^{(H)} \in \mathrm{Ind}_H^G(\xi)$ is defined by $f^{(H)}(g) = \begin{cases} \xi(g)^{-1} & g \in H \\ 0 & \text{o/w} \end{cases}$

then $\phi(f^{(H)}) = \dfrac{1}{(G:H)} v \neq 0$

As V is irreducible, we get $\mathrm{Image}(\phi) = V$

In particular $\dim(V) \leq \dim \mathrm{Ind}_H^G(\xi) = (G:H)$

(22.2) Example. $G = D_7$ dihedral group of size 14.

$H \simeq \mathbb{Z}/7\mathbb{Z}$ subgroup of G

Explicitly
$$\begin{array}{ccc} G & = & \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^7 = \mathrm{Id} \rangle \\ \cup & & \\ H & = & \langle \rho = s_1 s_2 \rangle \end{array}$$

$G \circlearrowright \mathbb{C}^2$ by
$$s_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$s_2 \longmapsto \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{bmatrix} \qquad \theta = \frac{2\pi}{7}$$

For $0 \leq k \leq 6$, let ξ_k be the irreducible finite-dim'l repn. of H given by
$$H \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$
$$\rho \longmapsto e^{ik\theta}$$

Claim: $\mathrm{Res}_H^G(\mathbb{C}^2) = \xi_1 \oplus \xi_6$

Proof. $\langle \xi_k, \mathrm{Res}_H^G(\mathbb{C}^2) \rangle =$ mult. of ξ_k in $\mathrm{Res}_H^G(\mathbb{C}^2)$

$$= \frac{1}{7} \sum_{\ell=0}^{6} e^{ik\ell\theta} \cdot \underbrace{2\cos(-\ell\theta)}_{\mathrm{Trace}_{\mathbb{C}^2}(\rho^{-\ell})}$$

$$= \frac{1}{7} \sum_{\ell=0}^{6} e^{ik\ell\theta} \left(e^{i\ell\theta} + e^{-i\ell\theta} \right)$$

$$= \frac{1}{7} \sum_{l=0}^{6} \left(e^{i(k+1) l\theta} + e^{i(k-1) l\theta} \right)$$

$$= \begin{cases} 1 & \text{if } k \equiv 1 \text{ or } -1 \mod 7 \\ 0 & \text{o/w} \end{cases}$$

$\square$

(22.3) Example. $G = S_5 \supseteq H = S_3 \times S_2$. Compute $\chi_{\mathrm{Ind}_H^G(\text{Trivial})}$

Recall (Cor. 21.2 page 4)

$$\chi_{\mathrm{Ind}_H^G(\text{Trivial})}(g) = \left| (G/H)^g \right|$$

$$S_5 / S_3 \times S_2 \longleftrightarrow \text{Set of 2-elt subsets of } \{1, \ldots, 5\} =: X_2$$

Conj. classes in $S_5 \leftrightarrow$ partitions of 5

$= \{(1^5), (2,1^3), (2,2,1), (3,1^2), (3,2), (4,1), (5)\}$

with # elts in conj. class resp. $\{1, 10, 15, 20, 20, 30, 24\}$

So $\chi_{\mathrm{Ind}_H^G(\text{Trivial})}(g)$ equals :

- $g = \mathrm{Id} \longmapsto |X_2^{\mathrm{Id}}| = \binom{5}{2} = 10$

- $g = (12) \longmapsto \left| X_2^{(12)} \right| = 1 + \binom{3}{2} = 4$
- $g = (12)(34) \longmapsto \left| X_2^{(12)(34)} \right| = 2$
- $g = (123) \longmapsto \left| X_2^{(123)} \right| = 1$
- $g = (123)(45) \longmapsto \left| X_2^{(123)(45)} \right| = 1$
- $g = (1234) \longmapsto 0$ and • $g = (12345) \longmapsto 0$

How many irreducible summands are there in $\mathrm{Ind}_{S_3 \times S_2}^{S_5}(\mathrm{Triv.})$?

$$\left(\chi_{\mathrm{Ind}_H^G \mathrm{Triv}}, \chi_{\mathrm{Ind}_H^G \mathrm{Triv}} \right) = \frac{1}{120} \left[(10)^2 + 10 \cdot (4)^2 + 15 (2)^2 + 20 + 20 \right]$$
$$= 3$$

$\Rightarrow \mathrm{Ind}_H^G(\mathrm{Triv})$ is a direct sum of 3 irred. repns. of S_5.

(22.4) $G = S_4$
$\quad\;\; \vee$
$\quad\;\; H = S_3$ (permutations fixing 4) $\qquad S_3 \circlearrowright \mathbb{C}^2 =: W$

$V = \mathrm{Ind}_H^G(W)$ is 8-dimensional.

$$\chi_W : \begin{array}{l} (1,1,1) \longmapsto 2 \\ (2,1) \longmapsto 0 \\ (3) \longmapsto -1 \end{array}$$

$$S_4 / S_3 = \mathrm{id}\, S_3 \sqcup (14)\, S_3 \sqcup (24)\, S_3 \sqcup (34)\, S_3$$

$\chi_V(e) = 8 = \dim V.$

$\chi_V((12))$: from the coset representatives id and (34) conjugate (12) to $(12) \in S_3$

$$\Rightarrow \chi_V((12)) = \chi_W((12)) + \chi_W((12)) = 0$$

Similarly we can compute

Partitions of 4	(1^4)	$(2,1^2)$	$(2,2)$	$(3,1)$	(4)
χ_V	8	0	0	-1	0

Comparing with the character table of S_4, we get

$$\chi_V = \chi_{V_1} + \chi_{V_2} + \chi_{V_3}$$ (see lecture 19 page 4)

$$V = V_1 \oplus V_2 \oplus V_3 .$$