# Lecture 23

(23.0) Let $G$ be a finite group and $H < G$ a subgroup

For $f \in \mathbb{C}H_{class}$ define $\mathrm{Ind}_H^G(f) : G \longrightarrow \mathbb{C}$

by
$$\mathrm{Ind}_H^G(f)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} f(x^{-1}gx) \qquad \forall\, g \in G$$

Lemma - $\mathrm{Ind}_H^G(f) \in \mathbb{C}G_{class}$

Proof.
$$\mathrm{Ind}_H^G(f)(\sigma g \sigma^{-1}) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}\sigma g \sigma^{-1} x \in H}} f(x^{-1}\sigma g \underbrace{\sigma^{-1}x}_{y = \sigma^{-1}x})$$

$$= \frac{1}{|H|} \sum_{\substack{y \in G \\ y^{-1}gy \in H}} f(y^{-1}gy) = \mathrm{Ind}_H^G(f)(g)$$

□

Thm (21.1) can be restated as

$$\chi_{\mathrm{Ind}_H^G(W)} = \mathrm{Ind}_H^G \chi_W .$$

## (23.1) Mackey's Theorem.

Let $H$ and $K$ be two subgroups of a finite group $G$.

Let $W$ be a f.d. repn. of $H$. Consider

$$V := \mathrm{Res}^G_K \, \mathrm{Ind}^G_H W$$

Define, for every $s \in G$,

$$\begin{tikzcd} K_s := K \cap sHs^{-1} \\ H_s := H \cap s^{-1}Ks \arrow[u, "\cong"', "\text{conjugation by } s := \mathrm{Ad}(s)"] \end{tikzcd}$$

Write $G = \bigsqcup_{j=1}^{\ell} K s_j H$. Then

Thm. $$\mathrm{Res}^G_K \circ \mathrm{Ind}^G_H = \bigoplus_{j=1}^{\ell} \mathrm{Ind}^K_{K_{s_j}} \circ \mathrm{Ad}(s_j^{-1}) \circ \mathrm{Res}^H_{H_{s_j}}$$

$$\begin{tikzcd} & G\text{-repns} \arrow[dr, "\mathrm{Res}^G_K"] & & \\ H\text{-repns} \arrow[ur, "\mathrm{Ind}^G_H"] \arrow[d, "\mathrm{Res}^H_{H_s}"'] & & K\text{-repns} \\ H_s\text{-repns} \arrow[r, "\mathrm{Ad}(s^{-1})"'] & K_s\text{-repns} \arrow[ur, "\mathrm{Ind}^K_{K_s}"'] & \end{tikzcd}$$

Proof. Let $H \circlearrowright W$ $[\pi : H \longrightarrow GL(W)]$ and $V = \mathrm{Res}^G_K \, \mathrm{Ind}^G_H (W)$. Let

$$V_j := \Big\{ \underset{\text{in } \mathrm{Ind}^G_H(W)}{f : G \longrightarrow W} \text{ s.t. } f(g) = 0 \ \forall \ g \notin K s_j H \Big\}$$

- $V_j$ is a subrepresentation of $V$ (as repns. of $K$).

Proof. Let $f \in V_j$ and $k \in K$.

$$(k \cdot f)(g) = f(k^{-1} g) = 0 \ \forall \ k^{-1} g \notin K s_j H$$

i.e. $(k.f)(g) = 0 \ \forall \ g \notin K s_j H$

$\Rightarrow k.f \in V_j$

- $V \simeq V_1 \oplus \dots \oplus V_\ell$ as $K$-representations (clear).

Claim: $V_j \simeq \mathrm{Ind}^K_{K_j} \circ \mathrm{Ad}(s_j^{-1}) \circ \mathrm{Res}^H_{H_j} (W)$

Given $f \in V_j$ ( i.e. $f : G \longrightarrow W$ such that

$$f(gh) = \pi(h)^{-1} f(g)$$
$$f(g) = 0 \ \forall \ g \notin K s_j H \ )$$

define $\tilde{f} : K \longrightarrow W$ by

$\tilde{f}(k) = f(k s_j)$. Let us verify that

$\tilde{f} \in \mathrm{Ind}_{K_j}^{K} \circ \mathrm{Ad}(s_j^{-1}) \circ \mathrm{Res}_{H_j}^{H} (W)$. Recall: $K_j = K \cap (s_j H s_j^{-1})$

$H_j = (s_j^{-1} K s_j) \cap H$

$$\begin{tikzcd}
K_j \arrow[r, "\mathrm{Ad}(s_j^{-1})"] & H_j \arrow[r, "\pi"] & GL(W) \\
x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & s_j^{-1} x s_j \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

To show: $\tilde{f}(k \cdot x) = (\pi(\mathrm{Ad}(s_j^{-1}))(x^{-1})) \cdot \tilde{f}(k)$

$\forall$ $k \in K$ and $x \in K \cap (s_j H s_j^{-1})$

if $x = s_j h s_j^{-1}$

then $\tilde{f}(k \cdot x) = f(k \cdot s_j \cdot h) = \pi(h)^{-1} f(k \cdot s_j)$

$= (\pi(\mathrm{Ad}(s_j^{-1}))(x^{-1})) \cdot \tilde{f}(k)$ as required.

Check that $f \longmapsto \tilde{f}$ is an iso. of $K$-repns.

[ $\widetilde{(k \cdot f)}(x) = (k \cdot f)(x s_j) = f(k^{-1} x s_j)$

$= (k \cdot \tilde{f})(x)$ ]

$\forall$ $k, x \in K$. $f \in V_j$

Injectivity. $\tilde{f} = 0 \iff f(ks_j) = 0 \ \forall k \in K$

$\iff f(ks_j h) = \pi(h)^{-1} f(ks_j) = 0 \ \forall k \in K, h \in H$

so $f(g) = 0 \ \forall g \in K s_j H$ and since $f \in V_j$,
$f(g) = 0 \ \forall g \notin K s_j H$. Hence $f = 0$.

Surjectivity. Given $\mathcal{F} : K \longrightarrow W$ satisfying

$$\mathcal{F}(k . x) = \pi(s_j^{-1} x s_j)\, \mathcal{F}(k) \quad \forall k \in K,\ x \in \underset{(s_j H s_j^{-1}) \cap K.}{\underset{\parallel}{K_{s_j}}}$$

define $f : G \longrightarrow W$ as:

$$f(g) = 0 \ \forall g \notin K s_j H \quad \text{and} \quad f(k s_j h) = \pi(h)^{-1} \cdot \mathcal{F}(k)$$
$$(\forall k \in K, h \in H)$$

Then by defn. $f \in \mathrm{Ind}_H^G(W)$ and $\tilde{f} = \mathcal{F}$.

□

(23.2) An example. $G = S_5$ $\quad H = S_3 \times S_2$
$K = S_4$

$K \backslash G / H$ has two elements

$$G = \underset{\substack{\uparrow \\ \sigma \in S_5 \text{ s.t.} \\ \sigma^{-1}(5) \in \{4,5\}}}{K . \mathrm{id} . H} \ \sqcup\ \underset{\substack{\uparrow \\ \sigma \in S_5 \text{ s.t.} \\ \sigma^{-1}(5) \in \{1,2,3\}}}{K . \overset{s}{\overset{\downarrow}{(15)}} . H}$$

Let $W = \mathbb{C}^2$ (viewed as a subspace of $\mathbb{C}^3$ consisting of vectors w/ sum of coordinates = 0).

$H = S_3 \times S_2 \circlearrowright \mathbb{C}^2$ ($S_3$ as usual, $S_2$ trivially).

Now $H_{(id)} = H \cap K = S_3 \times S_1 \times S_1$

$(= K_{(id)})$

$$\begin{aligned} H_s &= H \cap (sKs) = (S_3 \times S_2) \cap S_{\{5,2,3,4\}} \\ &= S_{\{2,3\}} \times S_{\{4,5\}} \\ K_s &= K \cap (sKs) = S_{\{1,2,3,4\}} \cap \left(S_{\{2,3,5\}} \times S_{\{1,4\}}\right) \\ &= S_{\{2,3\}} \times S_{\{1,5\}} \end{aligned}$$

conj by (15)

Thus by Mackey's Theorem

$$\mathrm{Res}^G_K \circ \mathrm{Ind}^G_H (W) \cong \left\{ \mathrm{Ind}^{S_4}_{S_2 \times S_2} (\text{sign} \oplus \text{trivial}) \right\}$$

$$\oplus \left\{ \mathrm{Ind}^{S_4}_{S_3} (\mathbb{C}^2) \right\}$$

(23.3) Compute $\left( \chi_{\mathrm{Ind}^G_K (\text{Trivial})} , \chi_{\mathrm{Ind}^G_H (\text{Trivial})} \right)$

$$= \dim \mathrm{Hom}_G \left( \mathrm{Ind}_K^G (\mathrm{Triv}) , \mathrm{Ind}_H^G (\mathrm{Triv}) \right)$$

$$\underset{\text{(Frobenius reciprocity)}}{=} \dim \mathrm{Hom}_K \left( \mathrm{Triv}, \mathrm{Res}_K^G \, \mathrm{Ind}_H^G (\mathrm{Triv}) \right)$$

$$= \dim \left( \mathrm{Ind}_H^G (\mathrm{Triv}) \right)^K \underset{\uparrow}{=} \left| K \backslash G / H \right|$$

[Problem # 6 of Set 7]