Symmmetric group S_n

Partitions of n

$$P(n) \ni \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$$
$$\lambda_1 + \cdots + \lambda_\ell = n$$

$$\mathrm{Irr}(S_n) \leftrightarrow P(n) \leftrightarrow \text{conjugacy classes in } S_n$$
$$\begin{array}{ccc} \Psi & & \Psi \\ V_\lambda & \leftrightarrow & \lambda \end{array}$$

$$L_\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(\text{Trivial}), \text{ where } S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$$
$$\| $$
$$\mathrm{Fun}(S_n / S_\lambda)$$

$$(\mathbb{C}S_n) \text{ class } \ni i_\lambda = \chi_{L_\lambda}$$
$$\Psi$$
$$\chi_\lambda = \chi_{V_\lambda}$$

Let $X = \{1, \cdots, n\}$

$$S_n / S_\lambda \leftrightarrow \mathrm{Part}(X ; \lambda)$$
$$\circlearrowright S_n \qquad \|$$
$$\{ X = X_1 \cup \cdots \cup X_\ell \text{ s.t. } |X_j| = \lambda_j \}$$

transitive

$$\circlearrowright S_n : \sigma(X_1 \cup \cdots \cup X_\ell) = \sigma(X_1) \cup \cdots \cup \sigma(X_\ell)$$

$$S_n \xrightarrow{\text{surj.}} \mathrm{Part}(X ; \lambda)$$
$$\Psi$$
$$\sigma \longmapsto \sigma\{1, \cdots, \lambda_1\} \cup \sigma\{\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2\} \cup \cdots$$

$$\mathrm{Stab}(\{1,\cdots,\lambda_1\}\cup\{\lambda_1+1,\cdots,\lambda_1+\lambda_2\}\cup\cdots) = S_\lambda$$

$$\begin{aligned}(i_\lambda, i_\mu) &= |S_\lambda \backslash S_n / S_\mu| \\ &= \#\text{ matrices with } \mathbb{Z}_{\geq 0} \text{ entries whose} \\ &\quad\ \text{row. sum} = \mu, \quad \text{col. sum} = \lambda\end{aligned}$$

$$S_n/S_\mu \longleftrightarrow \mathrm{Part}(X;\mu)$$

Let $Y_1 = \{1,\cdots,\lambda_1\}$, $Y_2 = \{\lambda_1+1,\cdots,\lambda_1+\lambda_2\}$, $\cdots$

$$\begin{array}{lcl} \mathrm{Part}(X,\mu) & & A \text{ where} \\ \quad\ \Psi & & a_{ij} = |X_i \cap Y_j| \\ X_1\cup\cdots\cup X_r & \longmapsto & \\ \mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0) & & \Rightarrow \displaystyle\sum_{j=1}^{n} a_{ij} = |X_i \cap Y| \\ & & \qquad\qquad = |X_i| = \mu_i \end{array}$$

e.g.

$$\begin{array}{cc} \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} & \begin{matrix} 3 \\ 2 \end{matrix} \\ \begin{matrix} 4 & \ \ 1 \\ \uparrow & \ \ \uparrow \end{matrix} & \end{array} \qquad \underbrace{\{1,2,5\}}_{=X_1}\cup\underbrace{\{3,4\}}_{=X_2}$$

$Y_1=\{1,2,3,4\}$ $Y_2=\{5\}$.

There's a (partial) order on $P(n)$:

$\lambda \geq \mu$ if $\lambda_1+\cdots+\lambda_j \geq \mu_1+\cdots+\mu_j$, $\forall j$

e.g. $(3,1,1,1)$ and $(2,2,2)$ are not comparable

(n) is the max

$(\underbrace{1,\cdots,1}_{n})$ is the min

$(n) \rightsquigarrow L_{(n)} = \mathrm{Ind}_{S_n}^{S_n}(\mathrm{Triv}) = \mathrm{Triv}$

$(n-1,1) \rightsquigarrow L_{(n-1,1)} = \mathrm{Ind}_{S_{n-1}}^{S_n}(\mathrm{Triv}) = \mathrm{Fun}(S_n/S_{n-1}) \simeq \mathbb{C}^n$

$= \mathrm{Triv} \oplus \underset{\substack{\uparrow \\ \text{standard}}}{\mathbb{C}^{n-1}}$

$$\boxed{L_\lambda = \bigoplus_{\mu \geq \lambda} V_\mu^{\oplus K_{\lambda\mu}}, \quad K_{\lambda\lambda} = 1}$$

$(5) \leftrightarrow \mathrm{Triv}$

$(4,1) \leftrightarrow$ 4-dim'l irr.

$(3,2) \leftrightarrow$ 5-dim'l irr.

$(3,1,1) \leftrightarrow$ 6-dim'l irr.

$\vdots$

$((3,1,1) \leq \lambda$ if $\lambda = (5)$ or $(4,1)$ or $(3,2))$

$K_{\lambda\mu}$ are explicitly computable (Frobenius–Schur)

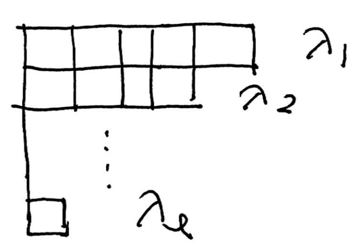
$\hookrightarrow$ Kotska numbers

e.g. $\mathrm{Ind}_{S_3}^{S_5}(\mathrm{Triv}) = \mathrm{Triv}^{\oplus n_1} \oplus (\mathbb{C}^4)^{\oplus n_2} \oplus (\mathbb{C}^5)^{\oplus n_3} \oplus \underset{\substack{\uparrow \\ \text{6-dim'l}}}{\underline{V}}$

(Young) Tableaux (YT)

$K_{\lambda\mu}$

$\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$



(3,2) $\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$

Semi-Standard T = a way of filling YT_λ

$$\begin{array}{c|l} & \leq \\ \hline \wedge & \end{array}$$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ (semi but <u>not</u> standard)

Standard : both strict

$$\begin{array}{c|l} & < \\ \hline \wedge & \end{array}$$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$

$K_{\lambda\mu}$ = # of ways of filling $YT_{\underline{\mu}}$ with
└ shape

type $\begin{cases} \lambda_1 & 1's \\ \lambda_2 & 2's \\ \lambda_3 & 3's \\ & \vdots \end{cases}$ and keep it semi-standard

e.g. $(3,1,1) \leq (5), (4,1), (3,2)$

$\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \\ \cline{1-1}\end{array}$

$$\underset{20}{\mathrm{Ind}_{S_3}^{S_5}(\mathrm{Triv})} = \underset{1}{\mathrm{Triv}} \oplus \underset{8}{(\mathbb{C}^4)^{\oplus 2}} \oplus \underset{5}{(5\text{-d})^{\oplus 1}} \oplus \underset{6}{(6\text{-dim})}$$

$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline\end{array}$

(5)

$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \\ \cline{1-1}\end{array}$

3 2
or 2 3

(4,1)

$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$

(3,2).

$\mu \in P(n)$

$\dim V_\mu = \chi_\mu(e) =$ # of times V_μ appears in $\mathbb{C}S_n$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

standard tableaux of shape μ $= K_{(1,\cdots,1),\mu} = \mathrm{Ind}_{S_1\times S_1\times\cdots}^{S_n}(\mathrm{Triv})$

$$n! = \sum_{\mu\vdash n} (\#\text{ standard YT of shape } \mu)^2$$

⟨Robinson - Schensted - Knuth⟩

$S_n \leftrightarrow$ Pair of standard YT of same shape

$M_{\lambda\mu} \leftrightarrow$ Pair of semi-standard YT of same shape (type λ & μ resp.)

$$(i_\lambda, i_\mu) = \left(\sum_\alpha \chi_\alpha K_{\lambda\alpha}, \sum_\beta \chi_\beta K_{\mu\beta}\right)$$

$$= \sum_\alpha K_{\lambda\alpha} K_{\mu\alpha}. \quad ((\chi_\alpha, \chi_\beta) = \delta_{\alpha\beta})$$

$$= (\text{Matrices w/ row sum} = \mu,\ \text{col. sum } \lambda)$$

⟨Frobenius Characteristic map⟩ (isometry)

$(\mathbb{C}S_n)_{\text{class}} \leftrightarrow$ Symm. poly's of degree n

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$

$$(f_1, f_2) = \frac{1}{n!}\sum_\sigma f_1(\sigma^{-1}) f_2(\sigma)$$

$P_k = x_1^k + x_2^k + \cdots \quad (k \geq 1)$

$P_\lambda = P_{\lambda_1} P_{\lambda_2} \cdots P_{\lambda_\ell} \quad (\lambda \vdash n)$

$$\delta_{C_\lambda} \longmapsto p_\lambda / z_\lambda$$

where $\delta_{C_\lambda}(g) = \begin{cases} 1 & \text{if } g \in C_\lambda \\ 0 & \text{o/w} \end{cases}$

$z_\lambda = |\text{centralizer of } C_\lambda|$

$i_\lambda = \chi_{\mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv})} \rightarrow$ complete symm. ftn

χ_λ – $\longmapsto$ Schur function

Complete symm. ftn

h_k = sum of all monomials of degree k

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$