Lecture 29 : Modules

(29.0) Throughout this lecture, the letter R will stand for an arbitrary ring ; while I will use A for a commutative ring.

(29.1) Left and right modules over R. – A left (resp. right) module M (resp. N) over R is an abelian group M (resp. N) together with a bilinear map $R\times M\to M$ (resp. $N\times R\to N$) such that

$$1\cdot m = m \qquad \left(\text{resp. } n\cdot 1 = n\right) \quad \forall\ m\in M$$
$$(a\cdot b)\cdot m = a\cdot(b m) \qquad \left(\quad n(a\cdot b) = (n a) b\right) \quad n\in N,\ a,b\in R$$

Note.– $(-a)\cdot m = -(a\cdot m) = a\cdot(-m)$ by bilinearity.

$$0_R\cdot m = 0_M \quad \forall\ m\in M.$$

(29.2) A more economical way of defining left/right modules over R would be to have an abelian group M (resp. N) and a ring hom $\lambda: R\to \mathrm{End}_{gp.}(M)$ (resp. $\rho: R^{op}\to \mathrm{End}_{gp}(N)$)

R^{op}: same as R as an ab. gp. $(a\cdot b)_{\text{in } R^{op}} = (ba)_{\text{in } R}$

where $\lambda(r): M\to M$, $m\mapsto r\cdot m$ (resp. $\rho(r): N\to N$, $n\mapsto n\cdot r$).

(29.3) Examples.– (i) $\mathfrak{a}\subset R$ left (right) ideal is a left (right) module $/R$.

(ii) Every abelian group is a module/$\mathbb{Z}$
(when the ring is commutative, left = right, just module).

(iii) $\forall\, n \geq 1$, $M = R^n$ (resp $N = R^n$) is a left (resp. right) module/R.

(29.4) Homomorphisms of modules.— Let M_1 and M_2 be two left R-modules. An R-linear map (or left R-module homomorphism) is a hom. of abelian groups $f: M_1 \to M_2$ such that $f(r.m_1) = r.f(m_1)$ $\forall\, r \in R,\ m_1 \in M_1$.

$\mathrm{Hom}_R(M_1, M_2)$ (= set of all R-linear maps $M_1 \to M_2$) has a structure of an abelian group.

We have the usual notions of submodules, quotient modules, kernels and images. We leave it to the reader to formulate and prove the analogous basic isomorphism results

(e.g. $f: M_1 \to M_2$ $\rightsquigarrow$
$$
\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, "\pi"'] & M_2 \\
M_1/\mathrm{Ker}(f) \arrow[r, "\cong", "\bar{f}"'] & \mathrm{Im}(f) \arrow[u, hook, "i"']
\end{tikzcd}
$$
)

(29.5) Direct sum of modules.

Let I be a set and $(M_i)_{i\in I}$ a set of (left) R-modules

$$\bigoplus_{i\in I} M_i = \left\{ (x_i)_{i\in I} : \begin{array}{l} x_i \in M_i \ \forall i\in I \\ x_i = 0 \text{ for all but finitely many } i\in I \end{array} \right\}$$

is again a (left) R-module (with componentwise operations:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i+y_i)_{i\in I}$$

$$r\cdot (x_i)_{i\in I} = (rx_i)_{i\in I}$$

Universal Property. Given a left R-module N and $\{f_i \in \mathrm{Hom}_R(M_i, N)\}_{i\in I}$ there exists a unique R-linear map

$$f : \bigoplus_{i\in I} M_i \longrightarrow N$$

$$(x_i)_{i\in I} \longmapsto \sum_{i\in I} f(x_i)$$

finite sum by defn of $\bigoplus_{i\in I} M_i$

(29.6) Let M be a left R-module. Let $M_1, M_2 \subset M$ be two submodules

Prop. $M \cong M_1 \oplus M_2$ iff

- $M_1 + M_2 = M$
- $M_1 \cap M_2 = (0)$

Proof. As $\begin{matrix} M_1 \hookrightarrow M \\ M_2 \hookrightarrow M \end{matrix}$ are R-linear, we get

by the universal property $M_1 \oplus M_2 \xrightarrow{f} M$.

Image of f = Submodule of M generated by M_1 and M_2

Kernel of f = $\{(x,-x) : x \in M_1 \cap M_2\}$

Thus f is an isomorphism iff $M = M_1 + M_2$ and $M_1 \cap M_2 = (0)$

□

Exercise. – Generalize to $\{M_i \hookrightarrow M\}_{i \in I}$. That is

$\bigoplus_{i \in I} M_i \to M$ is an isomorphism iff $M = \sum_{i \in I} M_i$ (submodule gen by $\{M_i\}_{i \in I}$)

and $M_i \cap \left(\sum_{j \neq i} M_j\right) = 0 \quad \forall i \in I$.

(29.7) Short exact sequences. – If M_1, M_2, M_3 are three left R-modules, and $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are R-linear maps; we say this sequence is exact (at M_2) if

Image of f = Kernel of g.

$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ short exact sequence

means. • f is injective • g is surjective

• $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Proposition. — A short exact sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is trivial* iff $\exists$ R-linear $s : M_3 \longrightarrow M_2$ s.t. $g \circ s = \mathrm{Id}_{M_3}$.

[*: i.e. we have (R-linear) isomorphism $M_2 \overset{\sim}{\underset{\eta}{\leftarrow}} M_1 \oplus M_3$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] & M_3 \arrow[r] & 0 \\
0 \arrow[r] & M_1 \arrow[u, equal] \arrow[r, hook, "i"] & M_1 \oplus M_3 \arrow[u, "\eta", "\cong"'] \arrow[r, "\pi"] & M_3 \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

commutes]

Proof. $(\Rightarrow)$ Take $j : M_3 \hookrightarrow M_1 \oplus M_3$ as ~~define~~ the usual inclusion and define $s : M_3 \longrightarrow M_2$ to be $\eta \circ j$.

$(\Leftarrow)$
$$\begin{aligned} M_1 \oplus M_3 &\longrightarrow M_2 \\ (x, y) &\longmapsto f(x) + s(y) \end{aligned}$$
is R-linear and it makes the diagram above commute.

Exercise. — Verify that η is an isomorphism.

(29.8) Direct product. – Again if I is a set and $\{M_i\}_{i\in I}$ a set of left R-modules, the direct product $\prod_{i\in I} M_i$ is defined to be:

$$\prod_{i\in I} M_i = \{ (x_i)_{i\in I} \text{ where } x_i \in M_i \ \forall\, i\in I \}$$

[Note: no finiteness condition]

Universal property. – Given a left R-module N and R-linear maps $f_i : N \longrightarrow M_i$, $\exists !$ R-linear map

$$N \longrightarrow \prod_{i\in I} M_i$$

$$x \longmapsto (f_i(x))_{i\in I}$$

[Note: this property will not be satisfied by direct sum, because of finiteness cond^{n}].

Remark. – For I, a finite set

$$\bigoplus_{i\in I} M_i \simeq \prod_{i\in I} M_i \quad \text{as (left) } R\text{-modules.}$$

(29.9) Tensor product. – Non commutative case.

Again let R be an arbitrary ring, and let M be a <u>right</u> R-module and N be a left R-module. Below we will define $M \underset{R}{\otimes} N$ – an abelian group.

Definition. – $M \underset{R}{\otimes} N$ is spanned (over $\mathbb{Z}$) by elements of the form $x \otimes y$ where $x \in M$ and $y \in N$. These (formal) linear combinations are subject to the following list of relations:

[Bilinearity]
$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \qquad \forall\ x, x_1, x_2 \in M$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \qquad y, y_1, y_2 \in N$$
$$r \in R$$

$$(x \cdot r) \otimes y = x \otimes (r \cdot y)$$

Universal Property. – There is a bilinear hom. (of abelian groups) $M \times N \xrightarrow{i} M \underset{R}{\otimes} N$ such that
$$i(x \cdot r, y) = i(x, ry) \quad \forall\ x \in M, y \in N, r \in R$$

Given any bilinear (over $\mathbb{Z}$) map $M \times N \xrightarrow{f} P$ ← another abelian gp

such that $f(x \cdot r, y) = f(x, r \cdot y)$,

$\exists!$ hom of abelian groups $\tilde{f} : M \underset{R}{\otimes} N \longrightarrow P$.

making the following diagram commute.

$$\begin{tikzcd} M \times N \arrow[r, "i"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, "\tilde{f}"] \\ & P \end{tikzcd}$$

Remark. – In general $M \otimes_R N$ has no other structure than that of an abelian group. If M (resp. N) is a bimodule [P is a R–R bimodule, if it is both left and right R-module s.t. $(r.x).s = r.(x.s)$ $\forall \begin{smallmatrix} x \in P \\ r,s \in R \end{smallmatrix}$]

then $M \otimes_R N$ has a structure of a left (resp. right) R-module

via :

$$r.(m \otimes n) = (r.m) \otimes n$$

$$\Big(\text{respectively, } (m \otimes n).r = m \otimes (n.r)\Big)$$

(29.10) Tensor product. – commutative case.

Now let A be a commutative ring. Since left and right modules are the same, we get

$$M, N : A\text{-modules} \rightsquigarrow M \otimes_A N : A\text{-module}$$

(same defn as on previous page)

Finer universal property of $M \otimes_A N$:—

(i) There exists an A-bilinear map $M \times N \xrightarrow{i} M \otimes_A N$

(ii) For any A-module P and an A-bilinear map

$$M \times N \xrightarrow{f} P$$

there exists a unique A-linear map $M \otimes_A N \xrightarrow{\tilde{f}} P$

making the following diagram commute.

$$\begin{tikzcd} M \times N \arrow[r, "i"] \arrow[dr, "f"'] & M \otimes_A N \arrow[d, "\tilde{f}"] \\ & P \end{tikzcd}$$

(29.11) Some exercises.- • $M \otimes_R R \simeq M$ $\forall$ right module over R, M

• $R \otimes_R N \simeq N$ for every left R-module N

• $\left(\bigoplus_{i \in I} M_i \right) \otimes_R N \simeq \bigoplus_{i \in I} (M_i \otimes_R N)$; for a set $\{M_i\}_{i \in I}$ of right R-modules and a left R-module N

• Similarly $M \otimes_R \left(\bigoplus_{i \in I} N_i \right) \simeq \bigoplus_{i \in I} M \otimes_R N_i$