Lecture 30

Localization of rings and modules

(30.0) Motivation. From number theory: Diophantine equations:

$P_1(x_1,\dots,x_m), \dots, P_n(x_1,\dots,x_m) \in \mathbb{Z}[x_1,\dots,x_n]$

Question:— find integer solutions to $P_1 = \dots = P_n = 0$.

Approach:— look for solutions over $\mathbb{Q}$, or $\underbrace{\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : \begin{array}{c} a,b\in\mathbb{Z} \\ b\neq 0 \\ p\nmid b \end{array} \right\}}_{\text{localizations of } \mathbb{Z} \text{ at prime ideals.}}$

and "patch the local solutions"

From geometry:— behaviour of a space near a point → localization!

(30.1) Let A be a commutative ring, and $S \subset A$.

We say S is <u>multiplicative closed</u> if $1\in S$ and $a,b\in S$ $\Rightarrow a\cdot b\in S$.

Consider the set $A\times S$ and introduce an equivalence relation: $(a,s)\sim(b,t)$ if there is $s'\in S$ such that $s'(at-bs)=0$

Claim:— This is an equivalence relation.

Proof:– Symmetry and reflexiveness are clear. Let us prove transitivity: $(a,s) \sim (b,t)$ and $(b,t) \sim (c,u)$

To prove:– $(a,s) \sim (c,u)$.

From the given relations, $\exists\ s', s'' \in S$ such that

$$s'at = s'bs \quad \text{and} \quad s''bu = s''ct$$

mult. closed

$$\text{Then} \quad (s's''t)\cdot au = s's''bsu$$
$$= (s's''t)\cdot cs \implies (\text{as } s's''t \in S)$$
$$(a,s) \sim (c,u)$$

$\square$

Definition:– The ring-of-fractions of A relative to S, denoted by $S^{-1}A$, is the set $A \times S/\sim$ with

- addition: $(a,s) + (a',s') = (as' + sa', ss')$
- Multiplication: $(a,s)\cdot(a',s') = (aa', ss')$
- Neutral elements: $0 = (0,1)$
  $1 = (1,1)$

Exercise:– Verify that addition and multiplication formulae given above are well-defined (i.e. descend to equivalence classes modulo $\sim$).

(30.2) Various remarks

(i) If A is an integral domain, the set of all non-zero elements of A forms a multiplicatively closed set. The corresponding ring of fractions is then a field, called the <u>field of fractions</u> of A; sometimes denoted by $Q(A)$.

eg. $A = \mathbb{Z}$, $Q(A) = \mathbb{Q}$

$A = K[x]$, $Q(A) = K(x)$ ← rational functions of one variable

(ii) For an arbitrary comm ring A, if one takes

S = set of non zero-divisors (of A)

Then S is multiplicatively closed and $S^{-1}A$ is often called the <u>total ring of fractions of A</u>. again denoted by $Q(A)$.

(iii) Note :— if $0 \in S$, $S^{-1}A = A \times S/_{\sim}$ is reduced to a single element ("zero ring" which we agreed to avoid)

Thus we will only consider multiplicatively closed sets not containing 0.

(iv) The definition of $S^{-1}A$ for (an integral domain A) first appeared in 1926 by H. Grell, a student of E. Noether.

(v) We often write the usual way for rational numbers:

$$\text{eq. class of } (a,s) \quad =: \quad \frac{a}{s} \qquad (a \in A,\ s \in S)$$

(30.3) Universal property:— (a) We have a natural ring homomorphism $j_S : \begin{array}{ccc} A & \longrightarrow & S^{-1}A \\ a & \longmapsto & \frac{a}{1} \end{array}$

such that for every $t \in S$, $j_S(t)$ is invertible in $S^{-1}A$ (its inverse being $\frac{1}{t}$).

(b) Let B be another comm. ring and let $f : A \to B$ be a ring homomorphism, such that $\forall\ t \in S$, $f(t) \in B$ is invertible. Then there exists a unique ring hom $\tilde{f} : S^{-1}A \to B$ making the following

diagram commute
$$\begin{tikzcd} A \arrow[r, "j_S"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, "\tilde{f}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & B \end{tikzcd}$$

(30.4) Lemma. — $\mathrm{Ker}(j_S) = \{a \in A : \exists\, s \in S \text{ with } s.a = 0\}$

Proof. — $j_S(a) = \dfrac{a}{1} = \dfrac{0}{1}$ if and only if

$\exists\, s \in S$ such that $s(a.1 - 0.1) = 0$, i.e. $sa = 0$ □

(30.5) Module of fractions. — Let M be an A-module and, as before, $S \subset A$ a multiplicatively closed set.

- Equivalence relation on $M \times S$: $(m,s) \sim (m',s')$ if there exists $t \in S$ such that
$$t(s'm - sm') = 0 \leftarrow \text{zero of } M.$$

- $S^{-1}M$ = module of fractions of M relative to S
$$= M \times S / \sim \quad \text{as a set}$$

→ addition $\dfrac{m}{s} + \dfrac{m'}{s'} = \dfrac{s'm + sm'}{ss'}$

→ Neutral elt. : $\dfrac{0}{1}$

$\to$ structure of a module over $S^{-1}A$ (and over A, via j_S)

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{a\cdot m}{st} \quad \forall\ a \in A,\ m \in M \quad s,t \in S.$$

Universal property :— (i) As a module over A, $\widetilde{M} := S^{-1}M$ has the property that $\rho(s) \in \mathrm{End}_{gp}(\widetilde{M})$ is invertible (here $\rho(s) : \widetilde{M} \to \widetilde{M}$, $x \mapsto s\cdot x$). We also have A-linear map $i_S : M \longrightarrow S^{-1}M$

(ii) Let N be another module over A, such that $\forall s \in S$ $\rho(s) \in \mathrm{End}_{gp}(N)$ is invertible. Given an A-linear map $f : M \to N$, $\exists!$ A-linear $\tilde{f} : \widetilde{M} \to N$ s.t.

$$\begin{tikzcd} M \arrow[r, "i_S"] \arrow[dr, "f"'] & \widetilde{M} \arrow[d, "\tilde{f}"] \\ & N \end{tikzcd}$$

commutes.

(30.6) Ideals in $S^{-1}A$.— Again let $S \subset A$ be a mult. closed subset of a comm. ring and $j_S : A \to S^{-1}A$ the natural ring hom. Given an ideal $\mathfrak{a} \subset A$, define

$S^{-1}\mathfrak{a}$ = ideal in $S^{-1}A$ generated by $j_S(\mathfrak{a})$.

Check. — ideal in $S^{-1}A$ generated by $j_S(\mathfrak{a})$ = module of fractions of $\mathfrak{a}$ (viewing $\mathfrak{a}$ as a module over A) rel. to S; so there is no abuse of notation in calling it $S^{-1}\mathfrak{a}$.

Proposition. — Every ideal in $S^{-1}A$ is of the form $S^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a} \subset A$.

[Note :— $S^{-1}\mathfrak{a} = S^{-1}A \iff S \cap \mathfrak{a} \neq \phi$]

Proof. — Let $\mathfrak{b} \subset S^{-1}A$ be an ideal. Consider $\mathfrak{a} = j_S^{-1}(\mathfrak{b})$.

In other words, $\mathfrak{a} = \left\{ a \in A : \cancel{\exists\, s \in S \text{ with } \tfrac{a}{s}}\ \ \frac{a}{1} \in \mathfrak{b} \right\}$

Claim. — $S^{-1}\mathfrak{a} = \mathfrak{b}$

Proof of the claim. — let $x \in \mathfrak{b}$. Then $x = \frac{y}{s}$ for some $y \in A$, $s \in S$

$$\Rightarrow \quad y = \frac{s}{1} \cdot \frac{y}{s} \in \mathfrak{b} \quad \Rightarrow y \in \mathfrak{a}$$

Thus $\mathfrak{b} \subset S^{-1}\mathfrak{a}$.

Conversely $a \in \mathfrak{a}$ and $s \in S$ $\Rightarrow$ $\frac{a}{s} = \underbrace{\frac{1}{s}}_{\in S^{-1}A}\ \underbrace{\frac{a}{1}}_{\in \mathfrak{b}} \in \mathfrak{b}$ (as it is an ideal)

$\Rightarrow S^{-1}\mathfrak{a} \subset \mathfrak{b}$.

$\square$

(30.7) **Proposition.**– Prime ideals in $S^{-1}A$ are of the form $S^{-1}\mathfrak{p}$ where $\mathfrak{p} \subsetneq A$ is a prime ideal s.t. $\mathfrak{p} \cap S = \phi$.

Proof. – Let $\mathfrak{q} \subsetneq S^{-1}A$ be a prime ideal. As in the proof of previous proposition, $\mathfrak{q} = S^{-1}\mathfrak{p}$ where $\mathfrak{p} = j_S^{-1}(\mathfrak{q})$ (recall: inverse image of a prime ideal is prime: Prop. 27.5 – page 4), is a prime ideal in A. $\mathfrak{p} \cap S = \phi$ since otherwise $S^{-1}\mathfrak{p} = S^{-1}A$ (see the note after the statement of prop. (30.6) previous page).

Now assume $\mathfrak{p} \subsetneq A$ is a prime ideal so that $\mathfrak{p} \cap S = \phi$.

Claim. — $S^{-1}\mathfrak{p}$ is prime in $S^{-1}A$.

Proof of the claim. - if $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}\mathfrak{p}$, we get

$$\frac{ab}{1} = \frac{(st)}{1} \cdot \frac{ab}{st} \in S^{-1}\mathfrak{p} \Rightarrow ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}$$

$$\Rightarrow \frac{a}{s} \in S^{-1}\mathfrak{p} \text{ or } \frac{b}{t} \in S^{-1}\mathfrak{p}.$$ Hence $S^{-1}\mathfrak{p}$ is prime. $\square$