Lecture 31

(31.0) Recall. – Last time we defined rings and modules of fractions.

A : commutative ring $\rightsquigarrow$ another comm. ring

$S \subset A$ multiplicatively closed set $\quad S^{-1}A$

M : A-module $\rightsquigarrow$ $S^{-1}M$: $S^{-1}A$-module

Last time we also proved:

(i) Every ideal of $S^{-1}A$ is of the form $S^{-1}\mathfrak{a}$, for some ideal $\mathfrak{a} \subset A$. (Prop. 30.6 page 7)

(ii) Prime ideals of $S^{-1}A$ $\longleftrightarrow$ Prime ideals of A meeting S trivially

$$\begin{array}{ccc} \psi & & \psi \\ \text{ideal in } S^{-1}A \text{ gen. by } j_S(\mathfrak{p}) = S^{-1}\mathfrak{p} & \longleftrightarrow & \mathfrak{p} \end{array}$$

(Prop. 30.7 page 8)

(31.1) Let $\mathfrak{p} \subsetneq A$ be a prime ideal and let $S = A \setminus \mathfrak{p}$. S being multiplicatively closed is equivalent to $\mathfrak{p}$ being prime.

$A_\mathfrak{p} := S^{-1}A$ is called the <u>localization</u> of <u>A at the prime ideal</u> $\mathfrak{p}$

Proposition. — $A_\mathfrak{p}$ is a local ring with the unique max'l ideal $\mathfrak{p}A_\mathfrak{p}$.

Proof. Let $\mathfrak{b} \subset A_\mathfrak{p}$ be an ideal. By Prop. (30.6) page 7, there is an ideal $\mathfrak{a} \subset A$ such that $\mathfrak{b} = S^{-1}\mathfrak{a}$, where $S = A \setminus \mathfrak{p}$.
If $\mathfrak{a} \cap S \neq \phi$, then $\mathfrak{b} = S^{-1}\mathfrak{a} = A_\mathfrak{p}$
Otherwise, $\mathfrak{a} \subset \mathfrak{p}$, hence $\mathfrak{b} \subset \mathfrak{p}A_\mathfrak{p}$ $(= S^{-1}\mathfrak{p})$. Thus every proper ideal of $A_\mathfrak{p}$ is contained in $\mathfrak{p} \cdot A_\mathfrak{p}$ which is thus the unique maximal ideal. □

(31.2) Modules of fractions and their homomorphisms.

Again let A be a commutative ring and $S \subset A$ a mult. closed set. Given two A-modules M and N and an A-linear map $f: M \to N$, we have an $S^{-1}A$-linear map

$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$$
$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

Proposition. — Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a short exact sequence of A-linear maps between

A-modules M_1, M_2 and M_3. Then the following sequence of $S^{-1}A$-linear maps is exact

$$0 \to S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \to 0$$

Proof. – (i) $\mathrm{Ker}(S^{-1}f) = (0)$. Let $\frac{m_1}{s} \in S^{-1}M_1$ be such that

$(S^{-1}f)\left(\frac{m_1}{s}\right) = 0$, i.e., $\frac{f(m_1)}{s} = 0$. Then $f(m_1) = s\cdot\left(\frac{f(m_1)}{s}\right) = 0$

$\Rightarrow m_1 \in \mathrm{Ker}(f) = (0)$. Hence $\mathrm{Ker}(S^{-1}f) = (0)$.

(ii) $S^{-1}(g)$ is surjective. Let $\frac{m_3}{s} \in S^{-1}M_3$. As g is surjective,

$\exists\, m_2 \in M_2$ s.t. $m_3 = g(m_2)$. Hence $\frac{m_3}{s} = S^{-1}g\left(\frac{m_2}{s}\right)$,

proving that $S^{-1}g$ is surjective.

(iii) $\mathrm{Ker}(S^{-1}g) = \mathrm{Im}(S^{-1}f)$.

$$\left(S^{-1}(g) \circ S^{-1}(f)\right)\left(\frac{m_1}{s}\right) \underset{\text{by defn.}}{=} \frac{g(f(m_1))}{s} = \frac{0}{s} = 0$$

$\Rightarrow \mathrm{Im}(S^{-1}f) \subset \mathrm{Ker}(S^{-1}g)$.

Conversely, if $\frac{m_2}{s} \in \mathrm{Ker}(S^{-1}g)$ we have $\frac{g(m_2)}{s} = 0$

$\Rightarrow g(m_2) = s\cdot\left(\frac{g(m_2)}{s}\right) = 0$. As $\mathrm{Ker}(g) = \mathrm{Im}(f)$,

$m_2 = f(m_1)$ for some $m_1 \in M_1$. This implies that

$$\frac{m_2}{s} = S^{-1}f\left(\frac{m_1}{s}\right) \Rightarrow \mathrm{Ker}(S^{-1}g) \subset \mathrm{Im}(S^{-1}f) \qquad \square$$

(31.3) Corollary. (1) Let $N \subset M$ be a submodule (over A).

Then $S^{-1}M / S^{-1}N \simeq S^{-1}(M/N)$.

(2) In particular, for ideal $\mathfrak{a} \subset A$

$$S^{-1}A / S^{-1}\mathfrak{a} \simeq S^{-1}(A/\mathfrak{a}) \simeq \bar{S}^{-1}(A/\mathfrak{a})$$

where $\bar{S}$ = image of S under $A \longrightarrow A/\mathfrak{a}$.

(31.4) Finiteness properties of rings. – Noetherian and Artinian rings.

In 1888, Kronecker published his findings on "ideal = product of prime ideals" research. He made a crucial assumption for ideals in polynomial rings. – that they are finitely generated. That this is true for ideals in polynomial rings, was proved later by Hilbert (Hilbert basis theorem). This property of rings (every ideal is finitely generated) has the following axiomatization

Definition. – A commutative ring A is called <u>Noetherian</u> if for every chain of ideals in A

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$$

[Ascending Chain Condition]

there is $k \geq 0$ such that $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \dots$

Theorem.– The following conditions on a comm. ring A are equivalent.

(i) A is Noetherian

(ii) Every non-empty set $\mathcal{S}$ of ideals of A has a max'l elt.

(iii) Every ideal $\mathfrak{a} \subset A$ is finitely generated.

Proof. – <u>(i) $\Rightarrow$ (ii)</u>. Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is not max'l, we have $\mathfrak{a}_1 \in \mathcal{S}$. s.t. $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1$. Continuing this fashion we get an ascending chain of ideals

$$\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \dots$$

By (i) this chain must terminate at $\mathfrak{a}_k$ which is then a max'l elt. of $\mathcal{S}$.

(ii) $\Rightarrow$ (iii) Let $\mathfrak{a}$ be an ideal. Consider the set

$$\mathcal{S} = \{ \underset{\text{ideal}}{\mathfrak{a}' \subsetneq \mathfrak{a}} \mid \mathfrak{a}' \text{ is finitely generated} \}$$

By (ii), this set has a max'l element, say $\tilde{\mathfrak{a}} \subset \mathfrak{a}$.

If $\tilde{\mathfrak{a}} \subsetneq \mathfrak{a}$, there is $a \in \mathfrak{a} \setminus \tilde{\mathfrak{a}}$ and hence

$(\tilde{\mathfrak{a}}, a) \in \mathcal{S}$ contradicts maximality of $\tilde{\mathfrak{a}}$.

Hence $\tilde{\mathfrak{a}} = \mathfrak{a} \Rightarrow \mathfrak{a}$ is finitely generated.

(iii) $\Rightarrow$ (i) Let $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$ be a chain of ideals in A.

Take $\mathfrak{a} = \bigcup_{j \geq 0} \mathfrak{a}_j \subset A$ still an ideal. By (iii)

$\mathfrak{a}$ is finitely generated, say by elements $a_1, \dots, a_n \in \mathfrak{a}$.

Now each $a_\ell \in \mathfrak{a}_{j_\ell}$ for some $j_\ell \geq 0$. Thus $\mathfrak{a} = \mathfrak{a}_j$

for $j = \max\{j_1, \dots, j_n\}$ and the chain terminates $\Rightarrow$

A is Noetherian. □

(31.5) Corollary. — Principal ideal domains are Noetherian

[e.g. $\mathbb{Z}$, $\mathbb{C}[x]$.]

(31.6) Corollary - Let $f : A \to B$ be a surjective ring hom of commutative rings. If A is Noetherian, so is B.

Localization of A (or ring of fractions $S^{-1}A$) is also Noetherian.

Proof. — Let $\mathfrak{b} \subset B$ be an ideal. $\mathfrak{a} = f^{-1}(\mathfrak{b}) \subset A$ is an ideal in a Noetherian ring, hence finitely generated; say $\mathfrak{a} = (a_1, \ldots, a_n)$. Then $\mathfrak{b} = (f(a_1), \ldots, f(a_n))$ is also finitely generated. $\square$

(31.7) Remark. Subring of a Noetherian ring need not be Noetherian. Take $R = \mathbb{C}[x_1, x_2, \ldots]$ polynomials in infinitely many variables.

R is not Noetherian (quite prototypical)

$\mathfrak{a}_1 = (x_1) \subset \mathfrak{a}_2 = (x_1, x_2) \subset \ldots$ never terminating ascending chain of ideals.

$R \hookrightarrow Q(R)$ = field of fractions of R (as R is a domain)

$\uparrow$

Noetherian (as it is a field).

(31.8) Hilbert Basis Theorem. — Let A be a Noetherian ring. Then $A[x]$ is a Noetherian ring.

Hence $\mathbb{Z}[x_1, \ldots, x_n]$, $K[x_1, \ldots, x_n]$ are Noetherian (i.e. every ideal is finitely generated).