Lecture 32

(32.0) Recall: last time we introduced localization of rings and modules and proved

Proposition.– Let M be a module over A (a commutative ring). Then the following are equivalent:

(i) $M = 0$

(ii) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subsetneq A$.

(iii) $M_{\mathfrak{m}} = 0$ for every max'l ideal $\mathfrak{m} \subsetneq A$.

Proof. – (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) because every maximal ideal is prime (Cor 27.3 p4). Now assume (iii) and assume $M \neq 0$. Let $x \in M$ be a non-zero element. Then

$$\mathrm{Ann}(x) := \{a \in A : ax = 0\} \subsetneq A$$

(proper ideal as $1 \notin \mathrm{Ann}(x)$)

Let $\mathfrak{m}$ be a max'l ideal of A containing $\mathrm{Ann}(x)$.

As $M_{\mathfrak{m}} = 0$ we get that there is an $s \in A \setminus \mathfrak{m}$ (mult-closed set used to define $M_{\mathfrak{m}}$)

s.t. $sx = 0$

But then we have $s \in \mathrm{Ann}(x) \setminus \mathfrak{m}$

contradicting the fact that $\mathrm{Ann}(x) \subset \mathfrak{m}$ $\square$

(32.1) Definition. — Let M be an A-module. We say M is Noetherian if it satisfies the ascending chain condition for its submodules:

for every chain of submodules $M_0 \subset M_1 \subset \cdots$

$\exists\ l \geq 0$ s.t. $M_l = M_{l+1} = \cdots$

We have the following analogue of Theorem (31.4)

Theorem. — The following are equivalent for an A-module M

(i) M is Noetherian

(ii) Every non-empty subset of submodules of M has a max'l element.

(iii) Every submodule of M is finitely generated.

The proof is exactly same as that of Thm. (31.4) and hence omitted.

(32.2) Corollary. — Let $0 \to M_1 \to M_2 \longrightarrow M_3 \longrightarrow 0$ be a short exact sequence of A-modules. Then M_2 is Noetherian if and only if M_1 and M_3 are.

Note: (1) A-module A is a Noetherian module if and only if A is a Noetherian ring

(2) Recall:— subring of a Noetherian ring need not be Noetherian.

(3) A finite direct sum of Noetherian modules is Noetherian. (Hint: induction on the number, using Cor(32.2) above).

(4) M : Noetherian A-mod $\Rightarrow$ $S^{-1}M$ is a Noetherian $S^{-1}A$-module ($S \subset A$ mult. closed set).

(32.3) Proposition.— Let A be a Noetherian ring and M an A-module. Then M is Noetherian iff M is finitely generated [i.e. $\exists\, x_1, \ldots, x_\ell \in M$ s.t. every $x \in M$ can be written (not necessarily uniquely) as $x = a_1x_1 + \ldots + a_\ell x_\ell$ where $a_1, \ldots, a_\ell \in A$]

Proof. ($\Rightarrow$) Clear by Thm 32.1 (iii)

($\Leftarrow$) As M is finitely generated, say by $x_1, \ldots, x_\ell \in M$; we get a surjective A-linear map

$$\underbrace{A \oplus \cdots \oplus A}_{\ell\text{-times}} \longrightarrow M$$

$$\underset{\text{from } i^{th} \text{ summand}}{a \in A} \longmapsto a x_i$$

A : Noetherian ring $\Rightarrow$ A is a Noetherian module (over A)

$\Rightarrow$ $\underbrace{A \oplus \dots \oplus A}_{\ell\text{-times}}$ is Noetherian. Hence so is M

using Cor. 32.2 (image of Noetherian is Noetherian). $\square$

(32.4) Examples. — (i) $A = K$ a field, $M =$ K-vector space.

M Noetherian $\iff$ $\dim_K M < \infty$.

(ii) $A = K[x]$ (Noetherian since PID), $M = K[x,y]$ is not Noetherian module over A (but Noetherian ring by itself).

(iii) An example of non-Noetherian ring.

$A =$ $\mathbb{C}$-valued fns. on $\mathbb{R}$. ; say continuous.

$F_n := \left[-\frac{1}{n}, \frac{1}{n}\right]$ $\quad n \geq 1$

$\mathfrak{a}_n = \{ f \in A \text{ s.t. } f|_{F_n} \equiv 0 \}$ is an ideal.

$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$ strictly increasing chain of ideals.

$\Rightarrow$ A is not Noetherian.

(32.5) Hilbert Basis Theorem. — Let A be a Noetherian ring. Then $A[x]$ is Noetherian.

Proof. – We will show that every ideal of $A[x]$ is finitely generated.

Let $\mathfrak{b} \subset A[x]$ be an ideal. For every $f(x) \in A[x]$ let $L.T.(f) \in A$ be the leading coefficient of f

$$f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \quad (a_n \neq 0) \implies LT(f) := a_n \in A.$$

[$LT(0)$ is defined to be 0].

Claim. — $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset A$ is an ideal

Proof of the claim. — $LT(f) + LT(g) \in \mathfrak{a}$. Because, if $\underset{k}{\deg(f)} \leq \underset{\ell}{\deg(g)}$,

$$\left.\begin{array}{l} x^{\ell-k} f \in \mathfrak{b} \\ g \in \mathfrak{b} \end{array}\right\} \implies LT(x^{\ell-k} f + g) \in \mathfrak{a}. \quad = LT(f) + LT(g).$$

Also $a \cdot LT(f) = LT(af) \in \mathfrak{a} \quad \forall\, a \in A,\ f \in \mathfrak{b}$.

As A is Noetherian, $\mathfrak{a} = (a_1, \dots, a_\ell)$ is finitely gen.

For each $j \in \{1, \ldots, \ell\}$, choose $f_j \in \mathfrak{b}$ s.t. $LT(f_j) = a_j$.

Consider the ideal $\mathfrak{b}' = (f_1, \ldots, f_\ell) \subset \mathfrak{b}$.

Let $r = \max\limits_{1 \leq j \leq \ell} \text{degree}(f_j)$. Let $M \subset A[x]$ be the A-submodule generated by $\{1, \ldots, x^{r-1}\}$ (i.e. M consists of polynomials of deg $< r$). Being f.g./Noeth., M is Noetherian. $\mathfrak{b} \cap M \subset M$ is an A-submodule of a Noetherian module hence finitely generated, say by $\{b_1, \ldots, b_k\}$.

Final Claim. — $\mathfrak{b} = (b_1, \ldots, b_k, f_1, \ldots, f_\ell)$

Proof. Let $f \in \mathfrak{b}$. If $\deg(f) < r$, then $f \in \mathfrak{b} \cap M$ and hence $f \in (b_1, \ldots, b_k)$.

Otherwise, let $a = LT(f)$; say $\deg(f_j) = d_j$

$\deg(f) = d \geq d_j \; \forall j$

if $a = r_1 a_1 + \ldots + r_\ell a_\ell$ (as $a \in \mathfrak{a} = (a_1, \ldots, a_\ell)$)

then $f - \sum r_j x^{d-d_j} f_j$ has smaller degree than f.

By induction we get that $f \in (b_1, \ldots, b_k, f_1, \ldots, f_\ell)$ $\square$