Lecture 33 Artinian Rings

(33.0) Definition. — Let A be a commutative ring. We say that A is Artinian (after Emil Artin) if for every chain of ideals

$$\mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots$$

there is $\ell \geq 0$ such that $\mathfrak{a}_\ell = \mathfrak{a}_{\ell+1} = \cdots$] Descending Chain Condition

Lemma. — Let $\mathfrak{S}$ be a set (non-empty) of ideals in an Artinian ring. Then $\mathfrak{S}$ has minimal elements.

Proof. — (see (i) $\Rightarrow$ (ii) of Theorem (31.4) p. 5)

Let $\mathfrak{a}_0 \in \mathfrak{S}$. If $\mathfrak{a}_0$ is minimal we are done. Otherwise we find $\mathfrak{a}_1 \in \mathfrak{S}$ s.t. $\mathfrak{a}_0 \supsetneq \mathfrak{a}_1$. As A is Artinian, this process must stop and we will arrive at a minimal element of $\mathfrak{S}$. $\square$

(33.1) Proposition. — Let A be an Artinian ring.

(a) Every prime ideal in A is maximal.

(b) There are only finitely many max'l ideals in A.

Proof — (a). Let $\mathfrak{p} \subsetneq A$ be a prime ideal. Then $A/\mathfrak{p}$ is an Artinian integral domain. We leave it to the reader to verify that the quotient ring of an Artinian ring is again Artinian.

Now take $x \neq 0$, $x \in A/p$. Consider the descending chain (in A/p)

$(x) \supset (x^2) \supset \cdots$ which eventually stabilizes

hence $\exists$ $k \geq \phi$ s.t. $x^k \in (x^{k+1})$ i.e. $x^k = y \cdot x^{k+1}$ for some $y \in A/p$. $\Rightarrow$ $x^k(1 - xy) = 0$. As A/p is a domain and $x \neq 0$, we get $xy = 1$, i.e. every non-zero elt. of A/p is invertible. Thus A/p is a field proving that p is a max'l ideal □

(b). Let $\mathcal{S}$ = the set of ideals which are finite intersections of max'l ideals of A.

$\mathcal{S} \neq \phi$ as max'l ideals exist and are in $\mathcal{S}$.

Let $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$ be a min'l elt. of $\mathcal{S}$ (see Lemma (33.0) above).

Claim. — Max'l ideals in A $= \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$

Pf of the claim. Let $\mathfrak{m} \subsetneq A$ be a max'l ideal. Then $\mathfrak{m} \cap \mathfrak{a} \in \mathcal{S}$ and is contained in $\mathfrak{a}$. By minimality of $\mathfrak{a}$, $\mathfrak{m} \cap \mathfrak{a} = \mathfrak{a}$.

By prop (28.2) page 2. $\mathfrak{m} \cap \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_\ell = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_\ell \subset \underset{\text{prime}}{\mathfrak{m}}$

$\Rightarrow \exists j$ s.t. $\mathfrak{m}_j \subset \mathfrak{m}$.

By maximality, $\mathfrak{m} = \mathfrak{m}_j$ and we are done. $\square$

(33.2) Corollary. — Let $\mathcal{N}$ = the ideal of A consisting of nilpotent elements. If A is Artinian, then

$\mathcal{N} = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_\ell$ (i.e. nilradical = Jacobson radical see HW10 for defns.)

From now on, we list the finitely many max'l ideals (= prime ideals) of A, (assumed to be Artinian) as $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$. Note that maximality $\Rightarrow$ coprime and we will freely use the following lemma, left as an exercise.

(33.3) Lemma. — Let R be a comm ring and $\mathfrak{a}, \mathfrak{b} \subset R$ be two coprime ideals. Then $\mathfrak{a}^j$ and $\mathfrak{b}^i$ are also coprime ($\forall j \geq 1$). Moreover $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$.

(33.4) Proposition. — The ideal $\mathcal{N} \subset A$ is nilpotent.

That is, there exists $n \geq 0$ s.t. $\mathcal{N}^n = (0)$.

Proof. — Consider the chain $\mathcal{N} \supset \mathcal{N}^2 \supset \dots$ As A is Artinian, we find n such that $\mathcal{N}^n = \mathcal{N}^{n+1} = \dots =: \mathfrak{a}$.

If $\mathfrak{a} \neq (0)$, then we consider $\mathcal{S}$ = set of all ideals $\mathfrak{b} \subset A$ s.t. $\mathfrak{a} \cdot \mathfrak{b} \neq (0)$

$\mathcal{S} \neq \phi$ since $\mathfrak{a}^2 = \mathfrak{a} \neq (0) \Rightarrow \mathfrak{a} \in \mathcal{S}$.

Pick a min'l element $I \in \mathcal{S}$. As $I \cdot \mathfrak{a} \neq (0)$, there is $x \in I$ s.t. $x\mathfrak{a} \neq (0) \Rightarrow (x) \in \mathcal{S}$. By minimality of I, $(x) = I$.

$(x\mathfrak{a})\mathfrak{a} = x\mathfrak{a}^2 \neq 0 \Rightarrow x\mathfrak{a} \subset (x)$ is also in $\mathcal{S}$.

Again by minimality $(x) = x\mathfrak{a} \Rightarrow \exists\, y \in \mathfrak{a}$ s.t. $x = xy$.

i.e. $x = xy = xy^2 = \dots$. But $y \in \mathfrak{a} \subset \mathcal{N}$ ← consists of nilpotent elts.

$\Rightarrow y^m = 0$ for some m

Hence $x = 0$ contradicting the choice of x. □

(33.5) Choose n such that $\mathcal{N}^n = (0)$. As the set of ideals $\{\mathfrak{m}_1^n, \dots, \mathfrak{m}_\ell^n\}$ are pairwise coprime, we get, by Theorem 26.6 page 5 ; we get a surjective map

$$\varphi : A \longrightarrow A/_{m_1^n} \times \cdots \times A/_{m_\ell^n} \quad \text{of rings.}$$

$\mathrm{Ker}(\varphi) = m_1^n \cdot \cdots \cdot m_\ell^n \subset \mathcal{N}^n = (0)$. Hence we have (see Lemma 33.3 above)

proved :

Theorem. — $A \simeq A/_{m_1^n} \times \cdots \times A/_{m_\ell^n}$. Each of $A/_{m_j^n}$ is a local ring with the unique max'l ideal $= \overline{m}_j$ (= image of m_j under $A \longrightarrow A/_{m_j^n}$).

Proof. — It remains to prove that $A/_{m_j^n}$ is local, $\forall\, 1 \leq j \leq \ell$.

Let $q \subsetneq A/_{m_j^n}$ be a prime ideal. Then $q =$ image of a prime ideal $p \subsetneq A$ containing $m_j^n \Rightarrow m_j = p$ (see Problem #1 of Set 11). □

(33.6) Now assume $(A, \mathfrak{m})$ is an Artinian local ring.

Proposition. — A is Noetherian.

Corollary (of this prop and Thm. 33.5) Artinian $\Rightarrow$ Noetherian

Proof of Proposition. — The proof depends on the following claim

Claim. — For each $j \geq 0$, $\mathfrak{m}^j / \mathfrak{m}^{j+1}$ is a finite-dim'l vector space over $k = A/\mathfrak{m}$.

Assuming this, let $\mathfrak{a} \subsetneq A$ be a proper ideal. Consider $\mathfrak{a}_j = \mathfrak{a} \cap \mathfrak{m}^j$

As $\dim_k \mathfrak{a}_j / \mathfrak{a}_{j+1} < \infty$, we get a finite basis of this vector space $\{a_1^{(j)}, \ldots, a_{d_j}^{(j)}\}$. Choose their lifts in $\mathfrak{a}_j$, denoted by the same symbol. Let $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ be the ideal generated by

$$\bigcup_{j \geq 1} \{a_1^{(j)}, \ldots, a_{d_j}^{(j)}\}$$ $\leftarrow$ finite set since if $\mathfrak{m}^n = (0)$, we get $\mathfrak{a}_m = 0$ $(\forall\, m \geq n)$

~~Now, by construction~~ $\cancel{\mathfrak{m} \cdot (\mathfrak{a}/\tilde{\mathfrak{a}}) = 0 \Rightarrow \mathfrak{a}/\tilde{\mathfrak{a}} = (0)}$

We leave it to the reader to verify (by decreasing induction) that $\tilde{\mathfrak{a}}_j = \mathfrak{a}_j \quad \forall\, \overset{1}{\cancel{0}} \leq j \leq n$ $[\tilde{\mathfrak{a}}_n = \mathfrak{a}_n = (0)$ base case$]$

Hence $\tilde{\mathfrak{a}}_1 = \tilde{\mathfrak{a}} = \mathfrak{a}_1 = \mathfrak{a}$ is finitely generated.

Thus every ideal in A is finitely generated and hence A is Noetherian.

Pf. of the claim. — View $k = A/m = (A/m^{j+1})/(m/m^{j+1})$.

Then subspaces (over A/m) in m^j/m^{j+1} correspond bijectively to ideals in A/m^{j+1} annihilated by m/m^{j+1} (i.e. $I \subset A/m^{j+1}$ s.t. $(m/m^{j+1}) \cdot I = 0$).

If $\dim_k (m^j/m^{j+1}) = \infty$, we can find an infinite strict descending chain of subspaces, and hence such a chain of ideals in A/m^{j+1} contradicting the definition of Artinian ring $\square$

(33.7) Hensel's Lemma. — Let (A, m) be an Artinian ring. Let $f(x) \in A[x]$ and assume we have $g(x), h(x) \in A[x]$ s.t.

- $f(x) - g(x)h(x) \in m[x]$
- $g(x)$ and $h(x)$ are coprime modulo m; i.e. $\exists\ a(x), b(x) \in A[x]$ s.t. $ag + bh \equiv 1$ in $(A/m)[x]$

Then $\exists\ \tilde{g}, \tilde{h} \in A[x]$ s.t. $f(x) = \tilde{g}(x) \cdot \tilde{h}(x)$ and
$$g(x) \equiv \tilde{g}(x),\ h(x) \equiv \tilde{h}(x) \pmod{m[x]}.$$

Proof. We will show the following, by induction on l :

$$\exists\ g_l, h_l \in A[x] \text{ s.t. } f - g_l h_l \in m^l[x]$$
$$\text{and } g_l - g \in m[x], \quad h_l - h \in m[x]$$

$l=1$ is the hypothesis of the lemma, i.e. $g_1 = g$; $h_1 = h$.

Assume we have constructed $\{g_l, h_l\}$ as above. Let

$$c_l(x) = f(x) - g_l(x) h_l(x) \in m^l[x].$$

Claim. – $g_{l+1} := g_l + c_l \cdot b$, $h_{l+1} := h_l + c_l \cdot a$ satisfy the listed requirements.

Pf. of the claim. – $f - g_{l+1} h_{l+1} = f - g_l h_l - c_l (a g_l + b h_l) + c_l^2\, a \cdot b$

$$= c_l (1 - a g_l - b h_l) + c_l^2\, a \cdot b$$

$$= \underbrace{c_l (1 - ag - bh)}_{\in m^{l+1}} + \underbrace{\underset{m^l[x]}{c_l}\, a\, \underbrace{(g - g_l)}_{\in m[x]}}_{m^{l+1}[x]} + c_l\, b\, (h - h_l) + c_l^2\, a b \quad \ldots$$

$\Rightarrow f - g_{l+1} h_{l+1} \in m^{l+1}[x]$ as required.

Finally $\quad g - g_{\ell+1} = \underbrace{g - g_\ell}_{\in \mathfrak{m}[x]} - \underbrace{c_\ell b}_{\substack{\in \mathfrak{m}^\ell[x] \\ \subset \mathfrak{m}[x]}} \in \mathfrak{m}[x]$

and similarly for $h - h_{\ell+1}$.

Thus, when $\ell = n$ s.t. $\mathfrak{m}^n = 0$, we arrive at the assertion of Hensel's Lemma. $\square$