(34.0) Recall that last time we defined an Artinian ring and proved that

(*) Artinian $\Rightarrow$ Noetherian and $\boxed{\text{dimension } 0}$ $\hookrightarrow$ i.e. every prime ideal is max'l.

Along the way we proved:

(i) Artinian domain = Field. Hence every prime ideal in an Artinian ring is max'l

(ii) N = ideal of nilpotent elements $= m_1 \cap \dots \cap m_\ell$ $= \prod_{j=1}^{\ell} m_j$. There are only finitely many max'l ideals $\{m_1, \dots, m_\ell\}$. <u>N is nilpotent</u>, i.e. $\exists\, n \geq 0$ s.t. $N^n = (0)$

(iii) $A \simeq A/m_1^n \times \dots \times A/m_\ell^n$. Each A/m_j^n is Artinian local ring.

(iv) Artinian local $\Rightarrow$ Noetherian.

Today we will prove the converse of this statement (*).

The proof is based on "primary decomposition".

(34.1) Definition. – An ideal $q \subsetneq A$ of a comm. ring is said to be <u>primary</u>, if for any $a, b \in A$,

$$a \cdot b \in q,\ b \notin q \implies a^n \in q \text{ for some } n \geq 1$$

Recall that radical of an ideal $\mathfrak{a} \subset A$ is defined as

$$r(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a} \text{ for some } n > 0\}$$

Lemma. – $q \subset A$ primary $\implies p = r(q)$ is prime.

Proof. – $a \cdot b \in p \implies a^k b^k \in q$ for some $k > 0$

$\implies$ either $b^k \in q$ or $(a^k)^n \in q$ for some $n > 0$

$\implies b \in r(q) = p$ or $a \in p$. Hence p is prime. $\square$

Examples. – (i) $A = k[x, y]$ (k : some field)

$q = (x, y^2)$ is primary but not power of a prime.

$p = r(q) = (x, y)$ and $p^2 \subsetneq q \subsetneq p$.

(ii) $A = K[x, y, z]/(xy - z^2) \supset p = (x, z)$

p is a prime ideal, but p^2 is not a primary ideal.

(34.2) Definition. — An ideal $\mathfrak{a} \subsetneq A$ is said to be *irreducible*, if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \Rightarrow \mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$
(for two ideals $\mathfrak{b}, \mathfrak{c} \subseteq A$)

Assume A is Noetherian from now on.

Lemma. — (i) Every ideal in A is a finite intersection of irreducible ideals.

(ii) Irreducible $\Rightarrow$ Primary.

Proof. — (i) Let Σ be the set of ideals of A which cannot be written as a finite intersection of irreducible ideals. If $\Sigma \neq \phi$, it must have a max'l elt, say $\mathfrak{a} \in \Sigma$. Then $\mathfrak{a}$ is not irreducible, hence

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \quad \text{for two ideals} \quad \begin{matrix} \mathfrak{a} \subsetneq \mathfrak{b} \\ \mathfrak{a} \subsetneq \mathfrak{c} \end{matrix}$$

Now $\mathfrak{b}, \mathfrak{c} \notin \Sigma$ as $\mathfrak{a}$ was max'l in Σ, implying

$$\begin{matrix} \mathfrak{b} = \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_k \\ \mathfrak{c} = \mathfrak{c}_1 \cap \dots \cap \mathfrak{c}_\ell \end{matrix} \Rightarrow \mathfrak{a} = \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_k \cap \mathfrak{c}_1 \cap \dots \cap \mathfrak{c}_\ell \notin \Sigma$$

[finite intersection of irreducible ideals] contradiction.

(ii) Let $\mathfrak{a} \subsetneq A$ be an irreducible ideal.

Working with $A/\mathfrak{a}$, we may assume (0) is an irred. ideal.

Let $x \cdot y \in (0)$, $y \notin (0)$ i.e. $xy = 0$, $y \neq 0$. To prove $x^n = 0$ for some $n > 0$.

Consider the chain $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \ldots$

[recall $\mathrm{Ann}(z) = \{a \in A \text{ s.t. } az = 0\} \underset{\text{ideal}}{\subset} A$]

By (ACC) for A (Noetherian), $\exists\, n > 0$ s.t. $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$

Claim. $(0) = (x^n) \cap (y)$

Pf. of the claim. — $a \in (y) \Rightarrow ax = 0$
$a \in (x^n) \Rightarrow a = bx^n$

So $bx^{n+1} = 0 \Rightarrow b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$

$\Rightarrow a = bx^n = 0$.

By irreducibility of (0), and $(y) \neq (0)$, we get $x^n = 0$ as required. $\square$

This lemma is usually referred to as "primary decomposition for Noetherian rings". We will return to it next time.

(34.3) Let $\mathfrak{a} \subsetneq A$ be an ideal in a Noetherian ring and $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_\ell$ be a primary decomposition of $\mathfrak{a}$. Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ $(1 \leq i \leq \ell)$ be corresponding prime ideals.

Lemma. – If $\mathfrak{p} \subsetneq A$ is a prime ideal minimal among the set of prime ideals containing $\mathfrak{a}$., then $\mathfrak{p} = \mathfrak{p}_i$ for some $i \in \{1, \dots, \ell\}$.

Proof. – $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_\ell \subset \mathfrak{p}$ implies (by Prop. 28.2 page 2) $\mathfrak{q}_i \subset \mathfrak{p}$ for some i. Hence, $\mathfrak{p}_i = r(\mathfrak{q}_i) \subset \mathfrak{p}$. Minimality of $\mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{p}_i$ $\square$

Cor. – There are only finitely many minimal prime ideals in a Noetherian ring.

(34.4) Theorem. Let A be a Noetherian ring of dim 0 (i.e. every prime ideal is maximal). Then A is Artinian.

Proof Let $m_1, \dots, m_\ell$ be minimal prime ideals of A (finitely many by previous corollary). By dim. 0 assumption, $m_1, \dots, m_\ell$ are max'l; hence the only max'l ideals of A.

Thus $N = \bigcap_{\mathfrak{p}:\text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{p}:\text{ min'l prime}} \mathfrak{p} = m_1 \cap \dots \cap m_\ell = \prod_{i=1}^{\ell} m_i$

Claim. $\exists\, n \geq 0$ s.t. $N^n = (0)$.

Arguing as in §33.5 (page 4), we get that

$$A \simeq A/m_1^n \times \dots \times A/m_\ell^n$$

Each A/m_i^n is a local (Noetherian) ring with only one prime ideal (= max'l ideal), namely m_i/m_i^n. The rest of the argument is entirely similar to that of Prop. (33.6)

Proof of the claim. As $N \subset A$ is f.g. and consists of nilpotent elements, $\exists\ x_1, \ldots, x_p \in N$, $n_1, \ldots, n_p \in \mathbb{Z}_{>0}$ s.t.

$x_j^{n_j} = 0$ and $N = \{ a_1 x_1 + \cdots + a_p x_p \mid a_1, \ldots, a_p \in A \}$
$(\forall\ 1 \leq j \leq p)$

Let $k = \max\{n_1, \ldots, n_p\}$ and $r > p(k-1)$.

Then $N^r = (0)$ as a typical element there is

$$\left(a_1^{(1)} x_1 + \cdots + a_p^{(1)} x_p\right) \cdots \left(a_1^{(r)} x_1 + \cdots + a_p^{(r)} x_p\right)$$

and each term must contain some x_j with exponent $\geq k$

and hence $= 0$. $\square$