Lecture 35

Primary decomposition

(35.0) Let A be a commutative Noetherian ring. Last time we proved:

Every proper ideal $\mathfrak{a} \subsetneq A$ can be written as a finite intersection $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_\ell$ where each $\mathfrak{q}_j$ is $\mathfrak{a}$ primary ideal. (Lemma 34.2 page 3).

e.g. $\mathfrak{a} = (x^2, xy) \subset K[x,y] = A$ (K: any field)

Let $\mathfrak{p}_1 = (x)$ $\mathfrak{p}_2 = (x,y)$ be (prime) ideals in A.

$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2 = \mathfrak{p}_1 \cap (x^2, y)$ two distinct primary decompositions

Today we will discuss uniqueness properties of the primary decomposition. We will keep $\mathfrak{a} \subsetneq A$ a proper ideal and $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_\ell$ fixed, and let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ $(1 \leq i \leq \ell)$ be the corresponding prime ideals.

(35.1) Assumptions. — (i) $\{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$ are distinct

(ii) $\forall\, i \in \{1, \dots, \ell\}$, $\mathfrak{q}_i \not\supset \bigcap_{\substack{1 \leq j \leq \ell \\ j \neq i}} \mathfrak{q}_j$

We say $\mathfrak{a} = \bigcap_{j=1}^{\ell} q_j$ is <u>reduced</u> if these assumptions are met. Note that (ii) is just saying that no q_i is redundant & can always be achieved by discarding redundant terms. We claim that (i) can always be achieved, starting from any primary decomposition. This is because of the following.

Lemma. — If $\tilde{q}_1, \ldots, \tilde{q}_n$ are primary with $r(\tilde{q}_j) = p$ $\forall\ 1 \leq j \leq n$; then $\tilde{q} = \bigcap_{1 \leq j \leq n} \tilde{q}_j$ is also primary and $r(\tilde{q}) = p$.

Proof. — $r(\tilde{q}) = \bigcap_{1 \leq j \leq n} r(\tilde{q}_j) = p$. Now if $xy \in \tilde{q}$ and $y \notin \tilde{q}$ then for some j, $y \notin \tilde{q}_j$ and $xy \in \tilde{q}_j \Rightarrow x \in p$ i.e. $x^m \in \tilde{q}$ for some $m > 0$. Hence $\tilde{q}$ is primary & $r(\tilde{q}) = p$. $\square$

From now on, we assume that the following is a <u>reduced</u> primary decomposition

$$\mathfrak{a} = q_1 \cap \cdots \cap q_\ell$$

$$\{p_i := r(q_i)\}_{1 \leq i \leq \ell} \leftarrow \text{corresponding distinct prime ideals.}$$

(35.2) **Lemma.** — Let $q \subsetneq A$ be primary and $p = r(q)$. For $x \in A$ we have the following.

(1) $x \in q \Rightarrow (q : x) = A$

(2) $x \notin q \Rightarrow (q : x)$ is primary and $r(q : x) = p$

(3) $x \notin p \Rightarrow (q : x) = q$

[recall: for two ideals $(\mathfrak{a} : \mathfrak{b}) := \{a \in A \mid a\mathfrak{b} \subset \mathfrak{a}\}$]

Proof. — (1) is obvious. For (3) if $y \in (q : x)$ and $x \notin p$ (i.e. $x^n \notin q\ \forall n > 0$), we get $xy \in q$ and $x^n \notin q\ (\forall n)$ $\Rightarrow y \in q$ as q is primary.

(2). We begin by proving $(q : x) \subset p$. Let $y \in (q : x)$, i.e. $xy \in q$. But $x \notin q$, so by defn. of primary ideal $y \in p$.

Thus by taking radicals $r(q : x) \subset p$.

$$r(q) = p \subset r(q : x)$$

We still have to prove that $(q : x)$ is primary. So, if $y.z \in (q : x)$ and $y^n \notin q\ (\forall n > 0)$, i.e. $y \notin p$; then

$$\left.\begin{array}{l} xyz \in q \\ y^n \notin q\ (\forall n) \end{array}\right\} \Rightarrow xz \in q \Rightarrow z \in (q : x)$$

□

(35.3) Theorem. — The set of prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$ is uniquely determined by $\mathfrak{a}$.

$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\} = \left\{ r(\mathfrak{a}:x) \;\middle|\; \begin{array}{l} x \in A \text{ s.t.} \\ r(\mathfrak{a}:x) \text{ is prime} \end{array} \right\}$$

no primary decomposition is needed to define this set.

Proof. — First we prove that each $\mathfrak{p}_j$ is of the form $r(\mathfrak{a}:x)$ for some $x \in A$. So, fix $j \in \{1, \dots, \ell\}$ and choose $x \in \bigcap_{i \neq j} \mathfrak{q}_i$ s.t. $x \notin \mathfrak{q}_j$ (exists since $\mathfrak{q}_j \not\supseteq \bigcup_{\substack{i \neq j \\ 1 \leq i \leq \ell}} \mathfrak{q}_i$).

Then $$(\mathfrak{a}:x) = \bigcap_{1 \leq i \leq \ell} (\mathfrak{q}_i : x) = (\mathfrak{q}_j : x)$$

$$\Rightarrow \quad r(\mathfrak{a}:x) = r(\mathfrak{q}_j : x) = \mathfrak{p}_j \text{ by Lemma (35.2) (2).}$$

Now, we prove the converse, i.e., assume $x \in A$ is such that $r(\mathfrak{a}:x)$ is prime. Then

$$(\mathfrak{a}:x) = \bigcap_{1 \leq i \leq \ell} (\mathfrak{q}_i : x) = \bigcap_{\substack{1 \leq i \leq \ell \\ \text{s.t. } x \notin \mathfrak{q}_i}} (\mathfrak{q}_i : x)$$

$$\Rightarrow r(\mathfrak{a}:x) = \bigcap_{\substack{1 \leq i \leq \ell \text{ s.t.} \\ x \notin \mathfrak{q}_i}} r(\mathfrak{q}_i : x) = \bigcap_{\substack{1 \leq i \leq \ell \\ \text{s.t. } x \notin \mathfrak{q}_i}} \mathfrak{p}_i \quad \left[\begin{array}{l}\text{again by Lemma} \\ 35.2\ (2)\end{array}\right]$$

Again by Prop 28.2 page 2 ; $r(\mathfrak{a}:x) = \mathfrak{p}_i$ for some $i \in \{1, \dots, \ell\}$. $\square$

(35.4) The set of prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$ is usually called "set of primes associated to $\mathfrak{a}$".

$$\text{Assoc}(\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$$ (depends only on $\mathfrak{a}$ by previous theorem)

Recall : Lemma 34.3 p. 5. — if $\mathfrak{p} \subsetneq A$ is a prime ideal minimal among those primes containing $\mathfrak{a}$, then $\mathfrak{p} \in \text{Assoc}(\mathfrak{a})$.

i.e. if $\text{Min}(\mathfrak{a}) := \left\{ \mathfrak{p} \subsetneq A \;\middle|\; \begin{array}{l} \mathfrak{p} \supset \mathfrak{a} \text{ and if} \\ \mathfrak{p} \supset \underset{\text{prime}}{\mathfrak{p}'} \supset \mathfrak{a}, \text{ then } \mathfrak{p}' = \mathfrak{p} \end{array} \right\}$

then $\text{Min}(\mathfrak{a}) \subset \text{Assoc}(\mathfrak{a})$. (Lemma 34.3 page 5)

Let us relabel so that $\text{Min}(\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ for some $k \leq \ell$.

Theorem. — $\{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$ are uniquely determined by $\mathfrak{a}$. More explicitly, $\forall\, i \in \{1, \dots, k\}$,

$$\mathfrak{q}_i = j_i^{-1}\left(j_i(\mathfrak{a}) \cdot A_{\mathfrak{p}_i}\right) \text{ where } j_i : A \longrightarrow A_{\mathfrak{p}_i}$$

Proof. – Let us keep $i \in \{1, \dots, k\}$ fixed and consider the ring hom. $j : A \longrightarrow A_{\mathfrak{p}_i}$ (recall: for a prime ideal $\mathfrak{p} \subsetneq A$; $A_{\mathfrak{p}} := \underbrace{(A \setminus \mathfrak{p})}_{\text{mult. closed set}}^{-1} \cdot A$).

Take $\mathfrak{b}$ = ideal in $A_{\mathfrak{p}_i}$ generated by $j(\mathfrak{a})$

$= S^{-1}\mathfrak{a}$ where $S = A \setminus \mathfrak{p}_i$

To show. – $j^{-1}(\mathfrak{b}) = \mathfrak{q}_i$

Since $\mathfrak{a} = \bigcap_{1 \le j \le l} \mathfrak{q}_j$, we get $S^{-1}\mathfrak{a} = \bigcap_{1 \le j \le l} S^{-1}(\mathfrak{q}_j)$

Now, for $j \neq i$, $S^{-1}\mathfrak{q}_j = S^{-1}A$ (in other words, $S \cap \mathfrak{q}_j \neq \phi$). This is because, otherwise we would have

$S \cap \mathfrak{q}_j = \phi$, i.e. $\mathfrak{q}_j \subset \mathfrak{p}_i \Rightarrow \underset{\overset{\|}{\mathfrak{p}_j}}{r(\mathfrak{q}_j)} \subset \mathfrak{p}_i$

$\mathfrak{p}_i$ was minimal, so $\mathfrak{p}_j = \mathfrak{p}_i$ contradicts assumption (i) (i.e. $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ are all distinct).

So we get $S^{-1}\mathfrak{a} = S^{-1}q_i$. Thus it remains to show that $j^{-1}(S^{-1}q_i) = q_i$. Clearly $q_i \subset j^{-1}(S^{-1}q_i)$ (Note: $q_i \subset p_i$ and $S = A \setminus p_i$; so $S \cap q_i = \phi$)

Conversely if $x \in j^{-1}(S^{-1}q_i) \subset A$, we get

$$\frac{x}{1} = \frac{a}{s} \quad \text{for some } a \in q_i \text{ and } s \in S$$

i.e. $s.x \in q_i$. Being primary, either $s^n \in q_i$ for some n or $x \in q_i$

But $s^n \in S$ $\forall n$ as S is multiplicatively closed and $S \cap q_i = \phi$. So, $x \in q_i$ as we wanted to prove. $\square$