Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $Y_h(\mathfrak{g})$, $U_q(L\mathfrak{g})$ the corresponding Yangian and quantum loop algebra, with deformation parameters related by $q = e^{\pi i \hbar}$. When $\hbar$ is not a rational number, we constructed in [11] a faithful functor $\Gamma$ from the category of finite–dimensional representations of $Y_h(\mathfrak{g})$ to those of $U_q(L\mathfrak{g})$. The functor $\Gamma$ is governed by the additive difference equations defined by the commuting fields of the Yangian, and restricts to an equivalence on a subcategory of $\text{Rep}_{fd}(Y_h(\mathfrak{g}))$ defined by choosing a branch of the logarithm. In this paper, we construct a tensor structure on $\Gamma$ and show that, if $|q| \neq 1$, it yields an equivalence of meromorphic braided tensor categories, when $Y_h(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ are endowed with the deformed Drinfeld coproducts and the commutative part of the $R$–matrix. The tensor structure arises from the abelian $q$KZ equations defined by a regularisation of the commutative $R$–matrix of $Y_h(\mathfrak{g})$.

**Contents**

1. Introduction .................................................. 2
2. Yangians and quantum loop algebras ......................... 9
3. The Drinfeld coproduct ....................................... 15
4. The commutative $R$-matrix of the Yangian ............... 23
5. The functor $\Gamma$ ........................................... 33
6. Tensor structure on $\Gamma$ .................................. 36
7. The commutative $R$–matrix of the quantum loop algebra 40
8. Kohno–Drinfeld theorem for abelian, additive $q$KZ equations 44

Appendix A. The inverse of the $T$–Cartan matrix of $\mathfrak{g}$ 49

References .................................................................. 53

---

VTL was supported in part through the NSF grant DMS–1206305.
1. Introduction

1.1. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, and $Y_h(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ the Yangian and quantum loop algebra of $\mathfrak{g}$. When the deformation parameter $h$ is not a rational number, so that $q = e^{\pi i h}$ is not a root of unity, we constructed in [11] an exact, faithful functor $\Gamma$ from the category of non-congruent representations of $Y_h(\mathfrak{g})$, a dense subcategory of $\text{Rep}_{\text{id}}(Y_h(\mathfrak{g}))$ (see 1.6 below), to the category of finite–dimensional representations of $U_q(L\mathfrak{g})$.

We proved moreover that

- $\Gamma$ induces an equivalence between the finite–dimensional representations of $U_q(L\mathfrak{g})$ and an explicit subcategory of those of $Y_h(\mathfrak{g})$.
- $\Gamma$ preserves the $q$–characters of Knight and Frenkel–Reshetikhin. In particular, for any $V, W \in \text{Rep}_{\text{id}}(Y_h(\mathfrak{g}))$ the classes of $\Gamma(V \otimes W)$ and $\Gamma(V) \otimes \Gamma(W)$ in the Grothendieck ring of $\text{Rep}_{\text{id}}(U_q(L\mathfrak{g}))$ are the same.

The main goal of this paper is to strengthen the latter result. Namely, we shall prove that the functor $\Gamma$ is compatible with the Drinfeld coproducts of $Y_h(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ and, when $|q| \neq 1$, that it yields an equivalence of meromorphic braided tensor categories when $\text{Rep}_{\text{id}}(Y_h(\mathfrak{g}))$ and $\text{Rep}_{\text{id}}(U_q(L\mathfrak{g}))$ are endowed with the meromorphic commutativity constraints defined by the commutative part of the $R$–matrix.

1.2. The Drinfeld coproduct on $U_q(L\mathfrak{g})$ was defined by Drinfeld in [5], and involves formal infinite sums of elements in $U_q(L\mathfrak{g})^{\otimes 2}$. Composing with the $\mathbb{C}^\times$–action on the first factor, Hernandez obtained a deformed coproduct, which is an algebra homomorphism

$$\Delta_\zeta : U_q(L\mathfrak{g}) \to U_q(L\mathfrak{g})((\zeta^{-1})) \otimes U_q(L\mathfrak{g})$$

where $\zeta$ is a formal variable [13, §6]. The map $\Delta_\zeta$ is coassociative, in the sense that $\Delta_{\zeta_1} \otimes 1 \circ \Delta_{\zeta_2} = 1 \otimes \Delta_{\zeta_1} \circ \Delta_{\zeta_2}$ [14, Lemma 3.2].

When computed on the tensor product of two finite–dimensional representations $\mathcal{V}_1, \mathcal{V}_2$ of $U_q(L\mathfrak{g})$, the deformed Drinfeld coproduct $\Delta_\zeta$ is analytically well–behaved in that the action of $U_q(L\mathfrak{g})$ on $\mathcal{V}_1((\zeta^{-1})) \otimes \mathcal{V}_2$ is the Laurent expansion at $\infty$ of a family of actions of $U_q(L\mathfrak{g})$ on $\mathcal{V}_1 \otimes \mathcal{V}_2$, whose matrix coefficients are rational functions of $\zeta$ [14, 3.3.2]. We denote $\mathcal{V}_1 \otimes \mathcal{V}_2$ endowed with this action by $\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2$.

1.3. In Section 3, we give simple contour integral formulae for the Drinfeld coproduct $\Delta_\zeta$, and therefore for the action of $U_q(L\mathfrak{g})$ on $\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2$. This yields an alternative proof of the rationality of $\otimes_\zeta$, as well as an explicit determination of its poles as a function of $\zeta$.

Specifically, let $\mathcal{V}$ be a finite–dimensional representation of $U_q(L\mathfrak{g})$, $\mathbf{I}$ the set of vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\Psi_i(z), \mathcal{X}_i^\pm(z)\}_{i \in \mathbf{I}}$ the $\text{End}(\mathcal{V})$–valued rational functions of $z \in \mathbb{P}^1$ whose Taylor expansion at $z = \infty$, 0 give the action of the generators of $U_q(L\mathfrak{g})$ on $\mathcal{V}$ (see Section 2.10), and $\sigma(\mathcal{V}) \subset \mathbb{C}^\times$ the set of poles of these functions.
Let \( \mathcal{V}_1, \mathcal{V}_2 \in \text{Rep}_{\text{id}}(U_q(L\mathfrak{g})) \), and let \( \zeta \in \mathbb{C}^\times \) be such that \( \zeta \sigma(\mathcal{V}_1) \) and \( \sigma(\mathcal{V}_2) \) are disjoint. Then, the action of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V}_1 \otimes_\zeta \mathcal{V}_2 \) is given by the following formulae for any \( m \in \mathbb{N} \) and \( k \in \mathbb{Z} \):

\[
\Delta_\zeta(\Psi_{i,\pm m}) = \sum_{p+q=m} \zeta^{\pm p} \Psi_{i,\pm p} \otimes \Psi_{i,\pm q},
\]

\[
\Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \int_{C_2} \Psi_i(\zeta^{-1}w) \otimes \mathcal{X}_i^+(w)w^{k-1}dw,
\]

\[
\Delta_\zeta(\mathcal{X}_{i,k}^-) = \int_{C_1} \mathcal{X}_i^-(\zeta^{-1}w) \otimes \Psi_i(w)w^{k-1}dw + 1 \otimes \mathcal{X}_{i,k}^-,
\]

where

- \( C_1, C_2 \subset \mathbb{C}^\times \) are Jordan curves which do not enclose 0.
- \( C_1 \) encloses \( \zeta \sigma(\mathcal{V}_1) \) and none of the points in \( \sigma(\mathcal{V}_2) \).
- \( C_2 \) encloses \( \sigma(\mathcal{V}_2) \) and none of the points in \( \zeta \sigma(\mathcal{V}_1) \).

1.4. The deformed Drinfeld coproduct \( \otimes_\zeta \) endows \( \text{Rep}_{\text{id}}(U_q(L\mathfrak{g})) \) with the structure of a meromorphic tensor category in the sense of [24]. This category is strict, in that for any \( \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \text{Rep}_{\text{id}}(U_q(L\mathfrak{g})) \), the identification of vector spaces

\[
(\mathcal{V}_1 \otimes_{\zeta_1} \mathcal{V}_2) \otimes_{\zeta_2} \mathcal{V}_3 = \mathcal{V}_1 \otimes_{\zeta_1 \zeta_2} (\mathcal{V}_2 \otimes_{\zeta_2} \mathcal{V}_3)
\]

intertwines the action of \( U_q(L\mathfrak{g}) \).

Meromorphic braided tensor categories were introduced by Soibelman in [24] to formalise the structure of the category of finite–dimensional representations of \( U_q(L\mathfrak{g}) \) endowed with the standard (Kac–Moody) tensor product and the \( R \)-matrix \( R(\zeta) \). The observation that such a structure also arises from the Drinfeld coproduct and the commutative part of the \( R \)-matrix (see §1.9–1.11 below) seems to be new.

1.5. In a related vein, a Drinfeld coproduct was defined for the double Yangian \( DY_h(\mathfrak{g}) \) by Khoroshkin–Tolstoy [19]. As for its counterpart for \( U_q(L\mathfrak{g}) \), it involves formal infinite sums. Moreover, the Yangian \( Y_h(\mathfrak{g}) \subset DY_h(\mathfrak{g}) \) is not closed under it.

By degenerating our contour integral formulae for \( \otimes_\zeta \), we obtain in Section 3.4 a family of actions \( V \otimes_s W \) of \( Y_h(\mathfrak{g}) \) on the tensor product of two finite–dimensional representations of \( Y_h(\mathfrak{g}) \), which is a rational function of a parameter \( s \in \mathbb{C} \). Its expansion at \( s = \infty \) should coincide with a deformation of the Drinfeld coproduct on \( DY_h(\mathfrak{g}) \) via the translation action of \( \mathbb{C} \) on \( Y_h(\mathfrak{g}) \), when the negative modes of \( DY_h(\mathfrak{g}) \) are reexpressed in terms of the positive ones through a Taylor expansion of the corresponding generating functions.

We also show that \( \otimes_s \) gives \( \text{Rep}_{\text{id}}(Y_h(\mathfrak{g})) \) the structure of a meromorphic tensor category, which is strict in that for any \( \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \text{Rep}_{\text{id}}(Y_h(\mathfrak{g})) \)

\[
(\mathcal{V}_1 \otimes_{s_1} \mathcal{V}_2) \otimes_{s_2} \mathcal{V}_3 = \mathcal{V}_1 \otimes_{s_1+s_2} (\mathcal{V}_2 \otimes_{s_2} \mathcal{V}_3)
\]
1.6. Before stating our main result, let us recall the notion of non–congruent representation of $Y_h(g)$ [11, §5.1]. Let $\{\xi_{i,r}, x^+_{i,r}, x^-_{i,r}\}_{i \in I, r \in \mathbb{N}}$ be the loop generators of $Y_h(g)$ (see [6], or §2 for definitions). Consider the generating series

$$\xi_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x^\pm_i(u) = h \sum_{r \geq 0} x^\pm_{i,r} u^{-r-1}$$

On a finite–dimensional representation $V$, these series are expansions at $u = \infty$ of $\text{End}(V)$–valued rational functions [11, Prop. 3.6]. $V$ is called non–congruent if, for any $i \in I$, the poles of $x^+_i(u)$ (resp. $x^-_i(u)$) do not differ by non–zero integers. If $V$ is non–congruent, the monodromy of the difference equations defined by the commuting fields $\xi_i(u)$ may be used to define an action of $U_q(Lg)$ on $\Gamma(V) = V$ [11].

1.7. If $V_1, V_2 \in \text{Rep}_{\mathbb{R}}(Y_h(g))$ are non–congruent, the Drinfeld tensor product $V_1 \otimes_s V_2$ is generically non–congruent in $s$. Our first main result is the following (see Theorem 6.2).

**Theorem.**

(i) There exists a meromorphic $GL(V_1 \otimes V_2)$–valued function $\mathcal{J}_{V_1,V_2}(s)$, which is natural in $V_1, V_2$ and such that

$$\mathcal{J}_{V_1,V_2}(s) : \Gamma(V_1) \otimes_{\mathbb{C}} \Gamma(V_2) \rightarrow \Gamma(V_1 \otimes_s V_2)$$

is an isomorphism of $U_q(Lg)$–modules, where $\zeta = e^{2\pi i s}$.

(ii) $\mathcal{J}$ is a meromorphic tensor structure on $\Gamma$. That is, for any non–congruent $V_1, V_2, V_3 \in \text{Rep}_{\mathbb{R}}(Y_h(g))$, the following is a commutative diagram

$$(\Gamma(V_1) \otimes_{\mathbb{C}} \Gamma(V_2)) \otimes_{\mathbb{C}} \Gamma(V_3) \longrightarrow \Gamma(V_1) \otimes_{\mathbb{C}} (\Gamma(V_2) \otimes_{\mathbb{C}} \Gamma(V_3))$$

where $\xi_i = \exp(2\pi i s_i)$.

1.8. Just as the functor $\Gamma$ is governed by the abelian, additive difference equations defined by the commuting fields $\xi_i(u)$ of the Yangian, the tensor structure $\mathcal{J}(s)$ arises from another such difference equation, namely an abelianisation of the $q$ KZ equations on $V_1 \otimes V_2$ [10, 23].
Specifically, let
\[ R_0(s) = 1 + \hbar \frac{\Omega_h}{s} + \cdots \]
be the diagonal part in the Gauss decomposition of the \( R \)-matrix of \( Y_\hbar(g) \) acting on \( V_1 \otimes V_2 \), where \( \Omega_h \in \mathfrak{h} \otimes \mathfrak{h} \) is the Cartan part of the Casimir tensor of \( g \) [19]. Unlike the analogous case of \( U_q(Lg) \) [18, 8], the expansion of \( R_0(s) \) does not converge near \( s = \infty \). We show, however, that \( R_0(s) \) possesses two distinct, meromorphic regularisations \( R_0^+, R_0^- \) in §4. These are asymptotic to \( R_0(s) \) for \( \pm \text{Re}(s/\hbar) \gg 0 \), and are related by the unitarity constraint
\[ R_0^+ R_0^- = 1. \]

Each \( R_0^+, R_0^- \) gives rise to the abelian \( q \)KZ equation
\[ \Phi^\pm(s+1) = R_0^\pm(s) \Phi^\pm(s) \]
where \( \Phi^\pm \) is an \( \text{End}(V_1 \otimes V_2) \)-valued function of \( s \). This equation admits a canonical right fundamental solution \( \Phi^{\pm,+}(s) \), which is holomorphic and invertible on an obtuse sector contained inside the half–plane \( \text{Re}(s) \gg 0 \), and possesses an asymptotic expansion of the form \( (1 + O(s^{-1}))e^{\frac{s \text{Re}(s/\hbar)}{2m}} \) within it (see Proposition 6.1). The tensor structure \( J_{V_1,V_2}(s) \) is then equal to one of \( \Phi^{\pm,+}(s+1)^{-1} \), which is a regularisation of the infinite product
\[ \cdots R_0^\pm(s+3) R_0^\pm(s+2) R_0^\pm(s+1) \]
Specifically,
\[ J_{V_1,V_2}(s) = e^{\hbar \gamma \Omega_h} \prod_{m \geq 1} R_0^\pm(s+m) e^{-\frac{\hbar m}{m}} \]
where \( \gamma = \lim_{n \to \infty} (1+1/2+\cdots+1/n-log(n)) \) is the Euler–Mascheroni constant.

1.9. As mentioned above, the regularisation of \( R_0(s) \) requires some work. A conjectural construction of \( R_0(s) \) as a formal infinite product with values in the double Yangian \( DY_\hbar(g) \) was given by Khoroshkin–Tolstoy [19, Thm. 5.2]. To make sense of this product, we notice in Section 4 that \( R_0(s) \) formally satisfies an abelian additive difference equation whose step is a multiple of \( \hbar \). We then prove that the coefficient matrix \( A(s) \) of this equation can be interpreted as a rational function of \( s \), and define \( R^0,\pm(s) \) as the canonical fundamental solutions of the difference equation. Let us outline this approach in more detail.

1.10. Let \( b_{ij} = d_i a_{ij} \) be the entries of the symmetrized Cartan matrix of \( g \). Let \( T \) be an indeterminate, and \( B(T) = ([b_{ij}]_T) \) the corresponding matrix of \( T \)-numbers. Then, there exists an integer \( l = mh^\vee \), which is a multiple of the dual Coxeter number \( h^\vee \) of \( g \), and is such that \( B(T)^{-1} = [l]^{-1}_T C(T) \).

\[^1\]This equation should in turn be a consequence of the (non–linear) difference equation satisfied by the full \( R \)-matrix of \( Y_\hbar(g) \) obtained from crossing symmetry.
where the entries of $C(T)$ are Laurent polynomials in $T$ with coefficients in $\mathbb{N}$ \cite{19}.\footnote{This result is stated without proof in \cite{19, p. 391}, and proved for $g$ simply–laced in \cite[Prop. 2.1]{15}. We give a proof in Appendix A, which also corrects the values of the multiple $m$ tabulated in \cite{19} for the $C_n$ and $D_n$ series. With those corrections, the value of $m$ for any $g$ is the ratio of the squared length of long roots and short ones.}

Consider the following $GL(V_1 \otimes V_2)$–valued function of $s \in \mathbb{C}$

$$A(s) = \exp \left( - \sum_{i,j \in I} \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} t_i^r(\xi_i) \otimes t_j \left( v + s + \frac{(l+r)\hbar}{2} \right) \right)$$

where

- $c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r$ are the entries of $C(T)$.
- the contour $C$ encloses the poles of $\xi_i(u)^{\pm 1}$ on $V_1$.
- $t_i(u) = \log(\xi_i(u))$ is defined by choosing a branch of the logarithm.
- $s \in \mathbb{C}$ is such that $v \rightarrow t_j(v + s + (l+r)\hbar/2)$ is analytic on $V_2$ within $C$, for every $j \in I$ and $r \in \mathbb{Z}$ such that $c_{ij}^{(r)} \neq 0$.

We prove in Section 4.5 that $A$ extends to a rational function of $s$ which has the following expansion near $s = \infty$

$$A(s) = 1 - l\hbar^2 \frac{\Omega_{\hbar}}{s^2} + O(s^{-3})$$

1.11. The infinite product $R_0^0(s)$ considered in \cite{19} formally satisfies

$$R_0^0(s + l\hbar) = A(s)R_0^0(s)$$

This difference equation is regular (that is, the coefficient of $s^{-1}$ in the expansion of $A(s)$ at $s = \infty$ is zero), and therefore admits two canonical meromorphic fundamental solutions $R_0^{0, \pm}(s)$. The latter are uniquely determined by the requirement that they be holomorphic and invertible for $\pm \text{Re}(s/\hbar) \gg 0$, and asymptotic to $1 + O(s^{-1})$ as $s \rightarrow \infty$ in that domain (see e.g., \cite[3, 20] or \cite[§4]{11}). Explicitly,

$$R_0^{0,+}(s) = \prod_{n \geq 0} A(s + n\hbar)$$

$$R_0^{0,-}(s) = \prod_{n \geq 1} A(s - n\hbar)$$

The functions $R_0^{0,\pm}(s)$ are distinct regularisations of $R_0^0(s)$, and are related by the unitarity constraint

$$R_{V_1,V_2}^0(s)R_{V_2,V_1}^0(-s)^2 = 1$$

We show in Theorem 4.9 that they define meromorphic commutativity constraints on $\text{Rep}_{\text{fd}}(Y_\hbar(g))$ endowed with the Drinfeld tensor product $\otimes_a$.\footnote{This result is stated without proof in \cite[19, p. 391]{19}, and proved for $g$ simply–laced in \cite[Prop. 2.1]{15}. We give a proof in Appendix A, which also corrects the values of the multiple $m$ tabulated in \cite{19} for the $C_n$ and $D_n$ series. With those corrections, the value of $m$ for any $g$ is the ratio of the squared length of long roots and short ones.}
1.12. Our second main result is a Kohno–Drinfeld theorem for the abelian, additive qKZ equations defined by $\mathcal{R}^{0,\pm}(s)$. Together with Theorem 1.7, it establishes an equivalence of meromorphic braided tensor categories between $\text{Rep}_{\text{id}}(Y_h(\mathfrak{g}))$ and $\text{Rep}_{\text{id}}(U_q(\mathfrak{L}))$ akin to the Kazhdan–Lusztig equivalence between the affine Lie algebra $\hat{\mathfrak{g}}$ and corresponding quantum group $U_q\mathfrak{g}$ [16, 17].

The abelian qKZ equations are the integrable system of difference equations for a meromorphic function $F : \mathbb{C}^n \to \text{End}(V_1 \otimes \cdots \otimes V_n)$, where $V_1, \ldots, V_n \in \text{Rep}_{\text{id}}(Y_h(\mathfrak{g}))$, which are given by

$$F(s+e_i) = A_i(s)F(s)$$

(1.1)

where $s = (s_1, \ldots, s_n)$, $\{e_i\}_{i=1}^n$ are the standard basis of $\mathbb{C}^n$, and

$$A_i(s) = \mathcal{R}_{i-1,i}^{0,\pm}(s_i - s - 1)^{-1} \cdots \mathcal{R}_{1,i}^{0,\pm}(s_1 - s - 1)^{-1} \cdot \mathcal{R}_{i,n}^{0,\pm}(s_i - s_n) \cdots \mathcal{R}_{i,i+1}^{0,\pm}(s_i - s_{i+1})$$

with $\mathcal{R}_{i,j}^{0,\pm} = \mathcal{R}_{V_i,V_j}^{0,\pm}$ [10, 23].

These equations admit a set of fundamental solutions $\Phi^\pm_\sigma$ which generalise the right/left solutions in the $n = 2$ case. They are parametrised by permutations $\sigma \in S_n$, and have prescribed asymptotic behaviour when $s_i - s_j \to \infty$ for any $i < j$, in such a way that $\text{Re}(s_{\sigma^{-1}(i)} - s_{\sigma^{-1}(j)}) \gg 0$.

By definition, the monodromy of (1.1) is the collection of meromorphic, $GL(V_1 \otimes \cdots \otimes V_n)$–valued functions of the variables $\zeta = e^{2\pi i s_i}$ given by $(\Phi^\pm_\sigma(s))^{-1} \cdot \Phi^\pm_\sigma(s)$.

1.13. A Kohno–Drinfeld theorem for the qKZ equations determined by the full (non–abelian) $R$–matrix of $Y_h(\mathfrak{g})$ was conjectured by Frenkel–Reshetikhin [10, §6], and states that the monodromy of (1.1) is given by the $R$–matrix of $U_q(\mathfrak{L})$ acting on a tensor product of suitable $q$–deformations of $V_1, \ldots, V_n$. This result was proved by Tarasov–Varchenko when $\mathfrak{g} = \mathfrak{sl}_2$, and $V_1, \ldots, V_n$ are evaluation representations with generic highest weights. It was later extended by the same authors to the $q$–difference equations determined by the $R$–matrix of $U_q(L\mathfrak{sl}_2)$ [25, 26].

One difficulty in addressing the general case is that, to the best our knowledge, no functorial way of relating arbitrary representations of $Y_h(\mathfrak{g})$ and $U_q(\mathfrak{L})$ was known to exist prior to [11], even for $\mathfrak{g} = \mathfrak{sl}_2$.

1.14. In the present paper, assuming that $|q| \neq 1$, we prove the Kohno–Drinfeld theorem for the abelian qKZ equations determined by $\mathcal{R}^{0,\pm}$ for any simple $\mathfrak{g}$ and $n$–tuple of representations of $Y_h(\mathfrak{g})$.

To this end, we first construct the commutative part $\mathcal{R}^0(\zeta)$ of the $R$–matrix of $U_q(\mathfrak{L})$ in §7 by following a procedure similar to that described in 1.9–1.11. Namely, we start from Damiani’s formula for $\mathcal{R}^0(\zeta)$ [4], show that if formally satisfies a regular $q$–difference equation with respect to the parameter $\zeta$, and deduce from this that it is the expansion at $\zeta = 0$ of the corresponding canonical solution (unlike the case of $Y_h(\mathfrak{g})$, no regularisation
of $\mathcal{R}^0(\zeta)$ is necessary here). We also show that $\mathcal{R}^0(\zeta)$ defines meromorphic commutativity constraints on $\text{Rep}_{\text{fd}}(U_q(Lg))$ endowed with the deformed Drinfeld coproduct.

We then prove the following (Theorem 8.5)

**Theorem.** Assume that $|q| \neq 1$, and let $\varepsilon \in \{\pm\}$ be such that $q^{\varepsilon \infty} = 0$. Let $V_1, \ldots, V_n \in \text{Rep}_{\text{fd}}(Y_h(\mathfrak{g}))$ be non–congruent, and let $V_i = \Gamma(V_i)$ be the corresponding representations of $U_q(Lg)$.

Then, the monodromy of the abelian $q$KZ equations determined by $\mathcal{R}^{0,\varepsilon}(s)$ on $V_1 \otimes \cdots \otimes V_n$ is given by $\mathcal{R}^0(\zeta)$. Specifically, the following holds for any $\sigma \in S_n$ and $i = 1 \ldots n - 1$ with $\sigma^{-1}(i) < \sigma^{-1}(i + 1)$,

$$
(\Phi^\varepsilon(\mathfrak{g}))^{-1} \cdot \Phi^\varepsilon_{i+1}(s) = \mathcal{R}^0_{V_1, V_{i+1}}(\zeta_{i+1}^{-1})
$$

The same result holds for the monodromy of the $q$KZ equations determined by $\mathcal{R}^{0,-\varepsilon}(s)$, provided $\mathcal{R}^0(\zeta)$ is replaced by $\mathcal{R}^{01}(\zeta^{-1})^{-1}$.

We shall prove the Kohno–Drinfeld theorem for the full (non–abelian) $q$KZ equations for any $\mathfrak{g}$ in a sequel to this paper [12].

1.15. We conjecture that the twist $J(s)$ also yields a non–meromorphic tensor structure on the functor $\Gamma$, when the categories $\text{Rep}_{\text{fd}}(Y_h(\mathfrak{g}))$ and $\text{Rep}_{\text{fd}}(U_q(Lg))$ are endowed with the standard monoidal structures arising from the Kac–Moody coproducts on $Y_h(\mathfrak{g}), U_q(Lg)$.

More precisely, the Drinfeld and Kac–Moody coproducts on $U_q(Lg)$ are related by a meromorphic twist, given by the lower triangular part $\mathcal{R}^{-U_q(Lg)}(\zeta)$ of the universal $R$–matrix [7]. A similar statement holds for $\text{Rep}_{\text{fd}}(\mathfrak{g})$ [12]. Composing, we obtain a meromorphic tensor structure $J(s)$ on $\Gamma$ relative to the standard monoidal structures

$$
\frac{\Gamma(V_1)(\zeta) \otimes \Gamma(V_2)}{J_{V_1, V_2}(s)} \xrightarrow{\mathcal{R}^{-U_q(Lg)}(\zeta)} \frac{\Gamma(V_1) \otimes \Gamma(V_2)}{J_{V_1, V_2}(s)}
$$

$$
\frac{\Gamma(V_1(s) \otimes V_2)}{J_{V_1, V_2}(s)} \xrightarrow{\mathcal{R}^{-Y_h(\mathfrak{g})}(s)} \frac{\Gamma(V_1(s) \otimes V_2)}{J_{V_1, V_2}(s)}
$$

We conjecture that $J_{V_1, V_2}(s)$ is holomorphic in $s$, and can therefore be evaluated at $s = 0$, thus yielding a tensor structure on $\Gamma$ with respect to the standard coproducts. We will return to this in [12].

1.16. The results of [11] hold for an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$. Although we restricted ourselves to the case of a finite–dimensional semisimple $\mathfrak{g}$ in this paper, our results on the Drinfeld coproducts of $Y_h(\mathfrak{g})$ and $U_q(Lg)$ are valid for an arbitrary $\mathfrak{g}$, and it seems likely that the same should hold for the construction of the tensor structure $J(s)$. The main obstacle in working in this generality is the construction and regularisation
of $R^0(s)$ for an arbitrary $g$. Once this is achieved, the proof of Theorems 1.7 and 1.14 carries over verbatim.

1.17. Outline of the paper. In Section 2, we review the definitions of $Y_\hbar(g)$ and $U_q(Lg)$. Section 3 is devoted to defining the Drinfeld coproducts on $U_q(Lg)$ and $Y_\hbar(g)$. We give a construction of the diagonal part $R^0$ of the $R$–matrix of $Y_\hbar(g)$ in §4. Section 5 reviews the definition of the functor $\Gamma$ given in [11]. The construction of a meromorphic tensor structure on $\Gamma$ is given in §6. In Section 7 we show that, when $|q| \neq 1$, the commutative part $\mathcal{R}^0$ of the $R$–matrix of $U_q(Lg)$ defines a meromorphic commutativity constraint on $\text{Rep}_{id}(U_q(Lg))$. Finally, in Section 8 we prove a Kohno–Drinfeld theorem for the abelian $q$KZ equations defined by $R^0$. Appendix A gives the inverses of all symmetrised $q$–Cartan matrices of finite type.

1.18. Acknowledgments. We are grateful to David Hernandez for his comments on an earlier version of this paper, to Sergey Khoroshkin for correspondence about the inversion of a $q$–Cartan matrix, and to Alexei Borodin and Julien Roques for correspondence on the asymptotics of solutions of difference equations. Part of this paper was written while the first author visited IHES in the summer of 2013. He is grateful to IHES for its invitation and wonderful working conditions.

2. Yangians and quantum loop algebras

2.1. Let $g$ be a complex, semisimple Lie algebra and $(\cdot, \cdot)$ the invariant bilinear form on $g$ normalised so that the squared length of short roots is 2. Let $h \subset g$ be a Cartan subalgebra of $g$, $\{\alpha_i\}_{i \in I} \subset h^*$ a basis of simple roots of $g$ relative to $h$ and $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ the entries of the corresponding Cartan matrix $A$. Set $d_i = (\alpha_i, \alpha_i)/2 \in \{1, 2, 3\}$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in I$.

2.2. The Yangian $Y_\hbar(g)$. Let $h \in \mathbb{C}$. The Yangian $Y_\hbar(g)$ is the $\mathbb{C}$–algebra generated by elements $\{x_{i,r}^\pm, \xi_{i,r}\}_{i \in I, r \in \mathbb{N}}$, subject to the following relations

(1) For any $i, j \in I$, $r, s \in \mathbb{N}$

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

(2) For $i, j \in I$ and $s \in \mathbb{N}$

$$[\xi_{i,0}, x_{j,s}^\pm] = \pm d_i a_{ij} x_{j,s}^\pm$$

(3) For $i, j \in I$ and $r, s \in \mathbb{N}$

$$[\xi_{i,r+1}, x_{j,s}^\pm] - [\xi_{i,r}, x_{j,s+1}^\pm] = \pm \hbar d_i a_{ij} \frac{1}{2} (\xi_{i,r}x_{j,s}^\pm + x_{j,s}^\pm \xi_{i,r})$$

(4) For $i, j \in I$ and $r, s \in \mathbb{N}$

$$[x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm \hbar d_i a_{ij} \frac{1}{2} (x_{i,r}^\pm x_{j,s}^\pm + x_{j,s}^\pm x_{i,r}^\pm)$$
(Y5) For $i,j \in I$ and $r,s \in \mathbb{N}$
\[ [x^+_{i,r}, x^-_{j,s}] = \delta_{ij} \xi_{i,r+s} \]

(Y6) Let $i \neq j \in I$ and set $m = 1 - a_{ij}$. For any $r_1, \ldots, r_m \in \mathbb{N}$ and $s \in \mathbb{N}$
\[ \sum_{\pi \in \mathcal{E}_m} [x^\pm_{i,r_{\pi(1)}}, [x^\pm_{i,r_{\pi(2)}}, \ldots, [x^\pm_{i,r_{\pi(m)}}, x^\pm_{j,s}] \ldots] = 0 \]

2.3. Remark. By [21, Lemma 1.9], the relation (Y6) follows from (Y1)–(Y3) and the special case of (Y6) when $r_1 = \cdots = r_m = 0$. In turn, the latter automatically hold on finite-dimensional representations of the algebra defined by relations (Y2) and (Y5) alone (see, e.g., [11, Prop. 2.7]). Thus, a finite-dimensional representation $V$ of $Y_h(\mathfrak{g})$ is given by operators $\{\xi_{i,r}, x^\pm_{i,r}\}_{i \in I, r \in \mathbb{N}}$ in $\text{End}(V)$ satisfying relations (Y1)–(Y5).

2.4. Assume henceforth that $h \neq 0$, and define $\xi_i(u), x_i^\pm(u) \in Y_h(\mathfrak{g})[[u^{-1}]]$
by
\[ \xi_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x_i^\pm(u) = h \sum_{r \geq 0} x_{i,r} u^{-r-1} \]

Proposition. [11] The relations (Y1),(Y2)–(Y3),(Y4),(Y5) and (Y6) are respectively equivalent to the following identities in $Y_h(\mathfrak{g})[[u, v; u^{-1}, v^{-1}]]$

(Y1) For any $i, j \in I$,
\[ [\xi_i(u), \xi_j(v)] = 0 \]

(Y2) For any $i, j \in I$,
\[ [\xi_{i,0}, x_j^\pm(u)] = \pm d_i a_{ij} x_j^\pm(u) \]

(Y3) For any $i, j \in I$, and $a = \hbar d_i a_{ij}/2$
\[ (u - v \mp a) \xi_i(u) x_j^\pm(v) = (u - v \mp a) x_j^\pm(v) \xi_i(u) \mp 2a x_j^\pm(u \mp a) \xi_i(u) \]

(Y4) For any $i, j \in I$, and $a = \hbar d_i a_{ij}/2$
\[ (u - v \mp a) x_i^\pm(u) x_j^\pm(v) \]
\[ = (u - v \pm a) x_j^\pm(v) x_i^\pm(u) + h \left( [x^\pm_{i,0}, x_j^\pm(v)] - [x_i^\pm(u), x^\pm_{j,0}] \right) \]

(Y5) For any $i, j \in I$
\[ (u - v) [x_i^\pm(u), x_j^\pm(v)] = -\delta_{ij} h (\xi_i(u) - \xi_i(v)) \]

(Y6) For any $i \neq j \in I$, $m = 1 - a_{ij}$, $r_1, \ldots, r_m \in \mathbb{N}$, and $s \in \mathbb{N}$
\[ \sum_{\pi \in \mathcal{E}_m} [x_i^\pm(u_{\pi(1)}), [x_i^\pm(u_{\pi(2)}), \ldots, [x_i^\pm(u_{\pi(m)}), x_j^\pm(v)] \ldots] = 0 \]
2.5. **Shift automorphism.** The group of translations of the complex plane acts on \( Y_h(g) \) by

\[
\tau_a(y_r) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} y_s
\]

where \( a \in \mathbb{C} \), \( y \) is one of \( \xi_i, x_i^\pm \). In terms of the generating series introduced in 2.4,

\[
\tau_a(y(u)) = y(u - a)
\]

Given a representation \( V \) of \( Y_h(g) \) and \( a \in \mathbb{C} \), set \( V(a) = \tau_a^*(V) \).

2.6. **Quantum loop algebra** \( U_q(Lg) \). Let \( q \in \mathbb{C}^\times \) be of infinite order. For any \( i \in I \), set \( q_i = q^{d_i} \). We use the standard notation for Gaussian integers

\[
[q]_q = \frac{q^n - q^{-n}}{q - q^{-1}}
\]

\[
[q]_q! = [q]_q[n-1]_q \cdots [1]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}
\]

The quantum loop algebra \( U_q(Lg) \) is the \( \mathbb{C} \)-algebra generated by elements \( \{ \Psi_{\iota,\pm \tau} \}_{\iota \in I, \tau \in \mathbb{N}}, \{ \chi_{i,k}^\pm \}_{i \in I, k \in \mathbb{Z}} \), subject to the following relations

(QL1) For any \( i, j \in I, r, s \in \mathbb{N} \),

\[
[\Psi_{i,\pm \tau}^\pm, \Psi_{j,\pm \tau}^\pm] = 0 \quad [\Psi_{i,\pm \tau}^\pm, \Psi_{i,\mp \tau}^\pm] = 0 \quad \Psi_{i,0}^\pm \Psi_{i,0}^- = 1
\]

(QL2) For any \( i, j \in I, k \in \mathbb{Z} \),

\[
\Psi_{i,0}^\pm \chi_{j,k}^\pm \Psi_{i,0}^- = q_i^{\pm a_{ij}} \chi_{j,k}^\pm
\]

(QL3) For any \( i, j \in I, \tau \in \{\pm\} \) and \( l \in \mathbb{Z} \)

\[
\Psi_{i,k+\tau}^\pm \chi_{j,l}^\pm - q_i^{\pm a_{ij}} \chi_{j,l+\tau}^\pm \Psi_{i,k}^\pm = q_i^{\pm a_{ij}} \chi_{i,k}^\pm \chi_{j,l+\tau}^\pm - \chi_{j,l+\tau}^\pm \chi_{i,k}^\pm
\]

for any \( k \in \mathbb{Z}_{\geq 0} \) if \( \tau = + \) and \( k \in \mathbb{Z}_{< 0} \) if \( \tau = - \)

(QL4) For any \( i, j \in I \) and \( k, l \in \mathbb{Z} \)

\[
\chi_{i,k+\tau}^\pm \chi_{j,l+1}^\pm - q_i^{\pm a_{ij}} \chi_{j,k+\tau}^\pm \chi_{i,l+1}^\pm = q_i^{\pm a_{ij}} \chi_{i,k+1}^\pm \chi_{j,l+\tau}^\pm - \chi_{j,l+\tau}^\pm \chi_{i,k+1}^\pm
\]

(QL5) For any \( i, j \in I \) and \( k, l \in \mathbb{Z} \)

\[
[\chi_{i,k+1}^+, \chi_{j,l+1}^-] = \delta_{ij} \frac{\Psi_{i,k+l}^+ - \Psi_{i,k+l}^-}{q_i - q_i^{-1}}
\]

where \( \Psi_{i,\mp k}^\pm = 0 \) for any \( k \geq 1 \).

(QL6) For any \( i \neq j \in I, m = 1 - a_{ij}, k_1, \ldots, k_m \in \mathbb{Z} \) and \( l \in \mathbb{Z} \)

\[
\sum_{\pi \in \mathfrak{S}_m} (-1)^s \left[ \begin{array}{c} m \cr s \end{array} \right] q_i^{a_{ij}} \chi_{i,k_{\pi(1)}}^\pm \cdots \chi_{i,k_{\pi(s)}}^\pm \chi_{j,d}^\pm \chi_{i,k_{\pi(s+1)}}^\pm \cdots \chi_{i,k_{\pi(m)}}^\pm = 0
\]
2.7. **Remark.** By [11, Lemma 2.12], the relation (QL6) follows from (QL1)--(QL3) and the special case of (QL6) when \( k_1 = \cdots = k_m = 0 \). In turn, the latter automatically hold on finite–dimensional representations of the algebra defined by relations (QL2) and (QL5) alone (see, e.g., [11, Prop. 2.13]). Thus, a finite–dimensional representation \( \mathcal{V} \) of \( U_q(Lg) \) is given by operators \( \{ \Psi_{i,\pm r}^\pm, \mathcal{X}_{i,k}^\pm \}_{i \in \mathbf{I}, r \in \mathbb{N}, k \in \mathbb{Z}} \) in \( \text{End}(\mathcal{V}) \) satisfying relations (QL1)--(QL5).

2.8. Define \( \Psi_i(z)^+, \mathcal{X}_i^\pm(z)^+ \in U_q(Lg)[[z^{-1}]] \) and \( \Psi_i(z)^-, \mathcal{X}_i^\pm(z)^- \in U_q(Lg)[[z]] \) by

\[
\Psi_i(z)^+ = \sum_{r \geq 0} \Psi_{i,r}^+ z^{-r} \quad \Psi_i(z)^- = \sum_{r \leq 0} \Psi_{i,r}^- z^{-r}
\]

\[
\mathcal{X}_i^\pm(z)^+ = \sum_{r \geq 0} \mathcal{X}_{i,r}^\pm z^{-r} \quad \mathcal{X}_i^\pm(z)^- = - \sum_{r < 0} \mathcal{X}_{i,r}^\pm z^{-r}
\]

**Proposition.** [11, Proposition 2.7] The relations (QL1), (QL2)--(QL3), (QL4), (QL5), (QL6) imply the following relations in \( U_q(Lg)[z, w; z^{-1}, w^{-1}] \)

- **(QL1)** For any \( i,j \in \mathbf{I} \), and \( h, h' \in h \),

\[
[\Psi_i(z)^+, \Psi_j(w)^+] = 0
\]

- **(QL2)** For any \( i,j \in \mathbf{I} \),

\[
\Psi_{i,0}^+ \mathcal{X}_i^\pm(z)^+ \left( \Psi_{i,0}^+ \right)^{-1} = q_i^{\pm a_{ij}} \mathcal{X}_j^\pm(z)^+
\]

- **(QL3)** For any \( i,j \in \mathbf{I} \)

\[
(z - q_i^{\pm a_{ij}} w) \Psi_i(z)^+ \mathcal{X}_j^\pm(w)^+
\]

\[
= (q_i^{\pm a_{ij}} z - w) \mathcal{X}_i^\pm(w)^+ \Psi_i(z)^+ - (q_i^{\pm a_{ij}} - q_i^{\mp a_{ij}}) q_i^{\pm a_{ij}} w \mathcal{X}_j^\pm(q_i^{\mp a_{ij}} z)^+ \Psi_i(z)^+
\]

- **(QL4)** For any \( i,j \in \mathbf{I} \)

\[
(z - q_i^{\pm a_{ij}} w) \mathcal{X}_i^\pm(z)^+ \mathcal{X}_j^\pm(w)^+ - (q_i^{\pm a_{ij}} z - w) \mathcal{X}_j^\pm(w)^+ \mathcal{X}_i^\pm(z)^+
\]

\[
= z \left( \mathcal{X}_{i,0}^\pm \mathcal{X}_j^\pm(w)^+ - q_i^{\pm a_{ij}} \mathcal{X}_j^\pm(w)^+ \mathcal{X}_{i,0}^\pm \right) + w \left( \mathcal{X}_{j,0}^\pm \mathcal{X}_i^\pm(z)^+ - q_i^{\pm a_{ij}} \mathcal{X}_i^\pm(z)^+ \mathcal{X}_{j,0}^\pm \right)
\]

- **(QL5)** For any \( i,j \in \mathbf{I} \)

\[
(z - w) [\mathcal{X}_i^\pm(z)^+, \mathcal{X}_j^\pm(w)^+] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left( z \Psi_i(w)^+ - w \Psi_i(z)^+ - (z - w) \Psi_{i,0}^- \right)
\]

- **(QL6)** For any \( i \neq j \in \mathbf{I} \), and \( m = 1 - a_{ij} \)

\[
\sum_{\pi \in S_m} \sum_{s=0}^{m} (-1)^s \left[ \begin{array}{c} m \\ s \end{array} \right] q_s \mathcal{X}_i^\pm(z_{\pi(1)})^+ \cdots \mathcal{X}_i^\pm(z_{\pi(s)})^+ \mathcal{X}_j^\pm(w)^+
\]

\[
\cdot \mathcal{X}_i^\pm(z_{\pi(s+1)})^+ \cdots \mathcal{X}_i^\pm(z_{\pi(m)})^+ = 0
\]
2.9. **Shift automorphism.** The group $\mathbb{C}^*$ of dilations of the complex plane acts on $U_q(Lg)$ by

$$
\tau_\alpha(Y_k) = \alpha^k Y_k
$$

where $\alpha \in \mathbb{C}^*$, $Y$ is one of $\Psi_i^\pm$, $\mathcal{X}_i^\pm$. In terms of the generating series of 2.8, we have

$$
\tau_\alpha(Y(z)^\pm) = Y(\alpha^{-1}z)^\pm
$$

Given a representation $\mathcal{V}$ of $U_q(Lg)$ and $\alpha \in \mathbb{C}^*$, we denote $\tau_\alpha^*(\mathcal{V})$ by $\mathcal{V}(\alpha)$.

2.10. **Rationality.** The following rationality property is due to Beck–Kac [1] and Hernandez [14] for $U_q(Lg)$ and to the authors for $Y_\hbar(g)$. In the form below, the result appears in [11].

**Proposition.**

(i) Let $V$ be a $Y_\hbar(g)$–module on which $\{\xi_{i,0}\}_{i \in I}$ acts semisimply with finite–dimensional weight spaces. Then, for every weight $\mu$ of $V$, the generating series

$$
\xi_i(u) \in \text{End}(V_\mu)[[u^{-1}]] \quad \text{and} \quad x_i^\pm(u) \in \text{Hom}(V_\mu, V_{\mu\pm\alpha_i})[[u^{-1}]]
$$

defined in 2.4 are the expansions at $\infty$ of rational functions of $u$. Specifically, let $t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} s_{i,0} \in Y_\hbar(g)$. Then,

$$
x_i^\pm(u) = 2d_i \hbar u^{-1} \left(\frac{\text{ad}(t_{i,1})}{u}\right)^{-1} x_i^\pm
$$

and

$$
\xi_i(u) = 1 + [x_i^+(u), x_i^-]
$$

(ii) Let $\mathcal{V}$ be a $U_q(Lg)$–module on which the operators $\{\Psi_{i,0}\}_{i \in I}$ act semisimply with finite–dimensional weight spaces. Then, for every weight $\mu$ of $\mathcal{V}$ and $\varepsilon \in \{\pm\}$, the generating series

$$
\Psi_i(z)^\pm \in \text{End}(V_\mu)[[z^{\mp1}]] \quad \text{and} \quad \mathcal{X}_i^\varepsilon(z)^\pm \in \text{Hom}(V_\mu, V_{\mu\pm\alpha_i})[[z^{\mp1}]]
$$

defined in 2.8 are the expansions of rational functions $\Psi_i(z), \mathcal{X}_i^\varepsilon(z)$ at $z = \infty$ and $z = 0$. Specifically, let $H_{i,\pm1}^\pm = \pm \Psi_{i,0}^\pm \mathcal{X}_i^\mp/(q_i - q_i^{-1})$. Then,

$$
\mathcal{X}_i^\varepsilon(z) = \left(1 - \varepsilon z \frac{\text{ad}(H_{i,1}^\pm)}{2_i z}\right)^{-1} \mathcal{X}_{i,0}^\varepsilon = z \left(1 - \varepsilon z \frac{\text{ad}(H_{i,-1}^\pm)}{2_i z}\right)^{-1} \mathcal{X}_{i,-1}^\varepsilon
$$

and

$$
\Psi_i(z) = \Psi_{i,0}^+ + (q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_i^-]
$$
2.11. Poles of finite–dimensional representations. By Proposition 2.10, we can define, for a given $V \in \text{Rep}_d(Y_h(\mathfrak{g}))$, a subset $\sigma(V) \subset \mathbb{C}$ consisting of the poles of the rational functions $\xi_i(u)^{\pm 1}, x_i^{\pm}(u)$.

Similarly, for any $V \in \text{Rep}_d(U_q(L\mathfrak{g}))$, we define a subset $\sigma(V) \subset \mathbb{C}^\times$ consisting of the poles of the functions $\Psi_i(z)^{\pm 1}, \chi_i^{\pm}(z)$.

2.12. The following is a direct consequence of Proposition 2.10 and contour deformation

**Corollary.**

(i) Let $V \in \text{Rep}_d(Y_h(\mathfrak{g}))$ and $\mathbb{C} \subset \mathbb{C}$ a Jordan curve enclosing $\sigma(V)$.

Then, the following holds on $V$ for any $r \in \mathbb{N}^4$

$$x_{i,r}^\pm = \frac{1}{\hbar} \oint_{\mathbb{C}} x_i^\pm(u)u^r du \quad \text{and} \quad \xi_{i,r} = \oint_{\mathbb{C}} \xi_i(u)u^r du$$

(ii) Let $V \in \text{Rep}_d(U_q(L\mathfrak{g}))$ and $\mathbb{C} \subset \mathbb{C}^\times$ a Jordan curve enclosing $\sigma(V)$ and not enclosing 0. Then, the following holds on $V$ for any $k \in \mathbb{Z}$ and $r \in \mathbb{N}^*$

$$\chi_{i,k}^\pm = \oint_{\mathbb{C}} \chi_i^\pm(z)z^{k-1}dz \
\Psi_{i,k}^{\pm} = \pm \oint_{\mathbb{C}} \Psi_i(z)z^{\pm r-1}dz$$

and

$$\oint_{\mathbb{C}} \Psi_i(u) \frac{dz}{z} = \Psi_{i,0}^+ - \Psi_{i,0}^-$$

2.13. The following result will be needed later.

**Lemma.** Let $V$ be a finite–dimensional representation of $Y_h(\mathfrak{g})$ and $i,k \in I$. If $u_0$ is a pole of $x_k^\pm(u)$, then $u_0 \pm \frac{hd_ia_{ik}}{2}$ are poles of $\xi_i(u)^{\pm 1}$.

**Proof.** Consider the relation (V3) of Proposition 2.4 and its inverse, as follows (here $b = \frac{hd_ia_{ik}}{2}$).

$$\text{Ad}(\xi_i(u))x_k^+(v) = \frac{u-v+b}{u-v-b} x_k^+(v) - \frac{2b}{u-v-b} x_k^+(u-b)$$

$$\text{Ad}(\xi_i(u))^{-1} x_k^+(v) = \frac{u-v-b}{u-v+b} x_k^+(v) + \frac{2b}{u-v+b} x_k^+(u+b)$$

Differentiating the first identity and using the fact that

$$\frac{d}{du} \text{Ad}(\xi_i(u))x_k^+(v) = \text{Ad}(\xi_i(u))[\xi_i(u)^{-1}\xi_i'(u),x_k^+(v)]$$

shows that

$$[\xi_i(u)^{-1}\xi_i'(u),x_k^+(v)] = \left( \frac{1}{u-v+b} - \frac{1}{u-v-b} \right) x_k^+(v)$$

$$+ \frac{1}{u-v-b} x_k^+(u-b) - \frac{1}{u-v+b} x_k^+(u+b) \quad (2.1)$$

3By a Jordan curve, we shall mean a disjoint union of simple, closed curves the inner domains of which are pairwise disjoint.

4we set \( \int_C f = \frac{1}{2\pi i} \int_C f \).
Thus, if \( x^+_k(v) \) has a pole at \( u_0 \) of order \( N \), then multiplying both sides by \( (v - u_0)^N \) and letting \( v \to u_0 \) we get:

\[
[\xi'_i(u), X] = \left( \frac{1}{u - u_0 + b} - \frac{1}{u - u_0 - b} \right) X
\]

where \( X = (v - u_0)^N x^+_k(u) \bigg|_{v = u_0} \). Hence the logarithmic derivative of \( \xi_i(u) \) has poles at \( u_0 \pm b \), which implies that \( u_0 \pm b \) must be poles of \( \xi_i(u)^\pm 1 \). The argument for \( x^-_k(v) \) is same as above, upon replacing \( b \) by \(-b\). \( \square \)

3. The Drinfeld coproduct

In this section, we review the definition of the deformed Drinfeld coproduct on \( U_q(L\mathfrak{g}) \) following [13, 14]. We then express it in terms of contour integrals, and use these to determine the poles of the coproduct as a function of the deformation parameter. By degenerating the integrals, we obtain a deformed Drinfeld coproduct for the Yangian \( Y_\hbar(\mathfrak{g}) \). We also point out that these coproducts define a meromorphic tensor product on the category of finite–dimensional representations of \( U_q(L\mathfrak{g}) \) and \( Y_\hbar(\mathfrak{g}) \).

3.1. Drinfeld coproduct on \( U_q(L\mathfrak{g}) \). Let \( \mathcal{V}, \mathcal{W} \in \text{Rep}_{fd}(U_q(L\mathfrak{g})) \). Twisting Drinfeld’s coproduct on \( U_q(L\mathfrak{g}) \) by the \( \mathbb{C}^\times \)–action on the first factor yields an action of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V}(1) \otimes \mathcal{W} \), where \( \zeta \) is a formal variable [13, 14]. This action is given on the generators of \( U_q(L\mathfrak{g}) \) by\(^5\)

\[
\Psi_{i,\pm m}^\pm \to \sum_{p+q=m} \zeta^{p+q} \Psi_{i,\pm p}^\pm \otimes \Psi_{i,\pm q}^\pm
\]

\[
\mathcal{X}_{i,k}^+ \to \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \sum_{l \geq 0} \zeta^{-l} \mathcal{X}_{i,-l} \otimes \mathcal{X}_{i,k+l}^+
\]

\[
\mathcal{X}_{i,k}^- \to \sum_{l \geq 0} \zeta^{k-l} \mathcal{X}_{i,k-l}^- \otimes \Psi_{i,l}^- + 1 \otimes \mathcal{X}_{i,k}^-^-
\]

Hernandez proved that the above formulae are the Laurent expansions at \( \zeta = \infty \) of a family of actions of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V} \otimes \mathcal{W} \) the matrix coefficients of which are rational functions of \( \zeta \) [14, 3.3.2].

3.2. Let \( \mathcal{V}, \mathcal{W} \in \text{Rep}_{fd}(U_q(L\mathfrak{g})) \) be as above, and \( \sigma(\mathcal{V}), \sigma(\mathcal{W}) \subset \mathbb{C}^\times \) their sets of poles (see 2.11). Let \( \zeta \in \mathbb{C}^\times \) be such that \( \zeta \sigma(\mathcal{V}) \) and \( \sigma(\mathcal{W}) \) are disjoint, and define an action of the generators of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V} \otimes \mathcal{W} \) as

\(^5\)We use a different convention than [13, 14]. The coproduct \( \Delta^{(H)}_\zeta \) given in [13, 14] yields an action on \( \mathcal{V} \otimes \mathcal{W}((\zeta)) \) obtained by twisting the Drinfeld coproduct by the \( \mathbb{C}^\times \)–action on the second tensor factor. The above action is equal to \( \Delta^{(H)}_{\zeta^{-1}}(\tau_\zeta(X)) \).
follows

\[ \Delta_\zeta(\Psi_{i,\pm m}) = \sum_{p+q=m} \zeta^{\pm p} \Psi_{i,\pm p} \otimes \Psi_{i,\pm q} \]

\[ \Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \oint_{C_2} \Psi_i(\zeta^{-1} w) \otimes \mathcal{X}_{i}^+(w) w^{k-1} dw \]

\[ \Delta_\zeta(\mathcal{X}_{i,k}^-) = \oint_{C_1} \mathcal{X}_{i}^-(\zeta^{-1} w) \otimes \Psi_i(w) w^{k-1} dw + 1 \otimes \mathcal{X}_{i,k}^- \]

where

- \( C_1, C_2 \subset \mathbb{C}^\times \) are Jordan curves which do not enclose 0.
- \( C_1 \) encloses \( \zeta \sigma(\mathcal{V}) \) and none of the points in \( \sigma(\mathcal{W}) \).
- \( C_2 \) encloses \( \sigma(\mathcal{W}) \) and none of the points in \( \zeta \sigma(\mathcal{V}) \).

The above operators are holomorphic functions of \( \zeta \in \mathbb{C}^\times \setminus \sigma(\mathcal{W})\sigma(\mathcal{V})^{-1} \).

The corresponding generating series \( \Delta_\zeta(\Psi_i(z)^\pm), \Delta_\zeta(\mathcal{X}_i^\sigma(z)^\pm) \) are the expansions at \( z = \infty, 0 \) of the \( \text{End}(\mathcal{V} \otimes \mathcal{W}) \)–valued holomorphic functions

\[ \Delta_\zeta(\Psi_i(z)) = \Psi_i(\zeta^{-1} z) \otimes \Psi_i(z) \]

\[ \Delta_\zeta(\mathcal{X}_i^+(z)) = \mathcal{X}_i^+(\zeta^{-1} z) \otimes 1 + \oint_{C_2} \frac{zw^{-1}}{z-w} \Psi_i(\zeta^{-1} w) \otimes \mathcal{X}_i^+(w) dw \]

\[ \Delta_\zeta(\mathcal{X}_i^-(z)) = \oint_{C_1} \frac{zw^{-1}}{z-w} \mathcal{X}_i^-(\zeta^{-1} w) \otimes \Psi_i(w) dw + 1 \otimes \mathcal{X}_i^-(z) \]

where the integrals are understood to mean the function of \( z \) defined for \( z \) outside of \( C_1, C_2 \). We shall prove below that their dependence in both \( \zeta \) and \( z \) is rational.

3.3. Theorem.

(i) The Laurent expansion of \( \Delta_\zeta \) at \( \zeta = \infty \) is given by the deformed Drinfeld coproduct of Section 3.1.

(ii) \( \Delta_\zeta \) defines an action of \( U_q(Lg) \) on \( \mathcal{V} \otimes \mathcal{W} \). The resulting representation is denoted by \( \mathcal{V} \otimes_\zeta \mathcal{W} \).

(iii) The action of \( U_q(Lg) \) on \( \mathcal{V} \otimes_\zeta \mathcal{W} \) is a rational function of \( \zeta \), with poles contained in \( \sigma(\mathcal{W})\sigma(\mathcal{V})^{-1} \).

(iv) The identification of vector spaces

\[ (\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2) \otimes_\zeta \mathcal{V}_3 = \mathcal{V}_1 \otimes_{\zeta_1\zeta_2} (\mathcal{V}_2 \otimes_\zeta_3 \mathcal{V}_3) \]

intertwines the action of \( U_q(Lg) \).

(v) If \( \mathcal{V} \cong \mathbb{C} \) is the trivial representation of \( U_q(Lg) \), then

\[ \mathcal{V} \otimes_\zeta \mathcal{W} = \mathcal{W} \quad \text{and} \quad \mathcal{W} \otimes_\zeta \mathcal{V} = \mathcal{W}(\zeta) \]

(vi) The following holds for any \( \zeta, \zeta' \in \mathbb{C}^\times \)

\[ \mathcal{V} \otimes_{\zeta\zeta'} \mathcal{W} = \mathcal{V}(\zeta) \otimes_{\zeta'} \mathcal{W} \]

and \( \mathcal{V}(\zeta') \otimes_\zeta \mathcal{W}(\zeta') = (\mathcal{V} \otimes_\zeta \mathcal{W})(\zeta') \).
(vii) The following holds for any \( \zeta \in \C^\times \)
\[
\sigma(V \otimes \zeta W) \subseteq (\zeta \sigma(V)) \cup \sigma(W)
\]

**Proof.** (i) Expanding \( \Delta_\zeta(\Psi_{i,m}^\pm) \) and \( \Delta_\zeta(\mathcal{X}_{i,k}^\pm) \) as Laurent series in \( \zeta^{-1} \) yields the following for any \( m \in \N \) and \( k \in \Z \)
\[
\Delta_\zeta(\Psi_{i,\pm m}^\pm) = \sum_{n=0}^{m} \zeta^{\pm n} \Psi_{i,\pm n}^\pm \otimes \Psi_{\pm (m-n)}^\pm
\]
\[
\Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \sum_{l \geq 0} \zeta^{-l} \mathcal{X}_i^{-l} \otimes \mathcal{X}_{i,k+l}^+
\]
\[
\Delta_\zeta(\mathcal{X}_{i,k}^-) = \sum_{l \geq 0} \zeta^{-l} \mathcal{X}_i^{-l} \otimes \mathcal{X}_{i,k+l}^-
\]
where the third and sixth equalities follow by Corollary 2.12, and the fourth by a change of variables.

(ii) By Remark 2.7, it suffices to check the relations (QL1)–(QL5). These follow from (i) and [13, Prop. 6.3], since it is sufficient to prove them when \( \zeta \) is a formal variable. Alternatively, a direct proof can be given along the lines of Theorem 3.5 below.

(iii) The rationality of \( V \otimes \zeta W \) follows from (i) and [14, 3.3.2]. Alternatively, let \( \{w_j\}_{j \in J} \subset \C^\times \) be the poles of \( \mathcal{X}_i^+(w) \) on \( W \), and
\[
\mathcal{X}_i^+(w) = \mathcal{X}_{i,0}^+ + \sum_{j \in J, n \geq 1} \mathcal{X}_{i,j,n}^+(w - w_j)^{-n}
\]
its corresponding partial fraction decomposition. Since \( C_2 \) encloses all \( w_j \), and \( \Psi_i(\zeta^{-1}w)w^{k-1} \) is regular inside \( C_2 \), we get
\[
\Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \sum_{j,n} \partial^{(n-1)} \left( \Psi_i(\zeta^{-1}w)w^{k-1} \right) \bigg|_{w=w_j}
\]
where \( \partial^{(p)} = \partial^p / p! \). This is clearly a rational function of \( \zeta \), whose poles are a subset of the points \( \zeta = w_j w_k'w^{-1} \), where \( w_k' \) is a pole of \( \Psi_i(w) \) on \( V \). A similar argument shows that \( \Delta_\zeta(\mathcal{X}_{i,k}^-) \) is also a rational function whose poles are contained in \( \sigma(W)\sigma(V)^{-1} \).

(iv) Follows from (i) and [14, Lemma 3.2].

(v), (vi) and (vii) are clear. \( \square \)
3.4. Drinfeld coproduct on \( Y_h(g) \). Let now \( V, W \in \Rep_{\fd}(Y_h(g)) \), and \( \sigma(V), \sigma(W) \subset \mathbb{C} \) their sets of poles. Let \( s \in \mathbb{C} \) be such that \( \sigma(V) + s \) and \( \sigma(W) \) are disjoint, and define an action of the generators of \( Y_h(g) \) on \( V \otimes W \) by

\[
\Delta_s(\xi_i(u)) = \xi_i(u - s) \otimes \xi_i(u)
\]
\[
\Delta_s(x^+_i(u)) = x^+_i(u - s) \otimes 1 + \oint_{C_2} \frac{1}{u - v} \xi_i(v - s) \otimes x^+_i(v) \, dv
\]
\[
\Delta_s(x^-_i(u)) = \oint_{C_1} \frac{1}{u - v} x^-_i(v - s) \otimes \xi_i(v) \, dv + 1 \otimes x^-_i(u)
\]

where

- \( C_2 \) encloses \( \sigma(W) \) and none of the points in \( \sigma(V) + s \).
- \( C_1 \) encloses \( \sigma(V) + s \) and none of the points in \( \sigma(W) \).
- The integrals are understood to mean the holomorphic functions of \( u \) they define in the domain where \( u \) is outside of \( C_1, C_2 \).

In terms of the generators \( \{\xi_{i,r}, x^+_i, x^-_i\} \), the above formulae read:

\[
\Delta_s(\xi_{i,r}) = \tau_s(\xi_{i,r}) \otimes 1 + h \sum_{p+q=r-1} \tau_s(\xi_{i,p}) \otimes \xi_{i,q} + 1 \otimes \xi_{i,r}
\]
\[
\Delta_s(x^+_i) = \tau_s(x^+_i) \otimes 1 + h^{-1} \oint_{C_2} \xi_i(v - s) \otimes x^+_i(v) v^r \, dv
\]
\[
\Delta_s(x^-_i) = h^{-1} \oint_{C_1} x^-_i(v - s) \otimes \xi_i(v) v^r \, dv + 1 \otimes x^-_i
\]

3.5. Theorem.

(i) The formulae in 3.4 define an action of \( Y_h(g) \) on \( V \otimes W \). The resulting representation is denoted by \( V \otimes_s W \).

(ii) The action of \( Y_h(g) \) on \( V \otimes_s W \) is a rational function of \( s \), with poles contained in \( \sigma(W) - \sigma(V) \).

(iii) The identification of vector spaces

\[
(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 = V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)
\]

intertwines the action of \( Y_h(g) \).

(iv) If \( V \cong \mathbb{C} \) is the trivial representation of \( Y_h(g) \), then

\[
V \otimes_s W = W \quad \text{and} \quad W \otimes_s V = W(s)
\]

(v) The following holds for any \( s, s' \in \mathbb{C} \),

\[
V \otimes_{s+s'} W = V(s) \otimes_{s'} W
\]

and \( V(s') \otimes_s W(s') = (V \otimes_s W)(s') \).

(vi) The following holds for any \( s \in \mathbb{C} \),

\[
\sigma(V \otimes_s W) \subset (s + \sigma(V)) \cup \sigma(W)
\]
Proof. (ii) is proved as in Theorem 3.3, and (iv)–(vi) are clear.

To prove (i), it suffices by Remark 2.3 to check that relations (Y1)–(Y5) hold on \( V \otimes s W \). By (v), we may assume that \( \sigma(V) \cap \sigma(W) = \emptyset \), and that \( s = 0 \). We choose the contours \( C_1 \) and \( C_2 \) enclosing \( \sigma(V) \) and \( \sigma(W) \) respectively, such that they do not intersect. The relation (Y1) holds trivially. The relations (Y2) and (Y3) are checked in 3.6, (Y4) in 3.7 and (Y5) in 3.8.

(iii) is proved in 3.9. \( \square \)

3.6. **Proof of (Y2) and (Y3).** We prove these relations for the + case only. By Proposition 2.4, it is equivalent to show that \( \Delta_0 \) preserves the relation

\[
\xi_i(u_1) x_j^+(u_2) \xi_i(u_1)^{-1} = \frac{u_1 - u_2 + a}{u_1 - u_2 - a} x_j^+(u_2) - \frac{2a}{u_1 - u_2 - a} x_j^+(u_1 - a)
\]

where \( a = \hbar d_i a_{ij} / 2 \). It suffices to prove this for \( u_1, u_2 \) large enough, and we shall assume that \( u_2 \) lies outside of \( C_2 \), and that \( u_1 \) lies outside of \( C_2 + a \).

Applying \( \Delta_0 \) to the left–hand side gives

\[
\xi_i(u_1) x_j^+(u_2) \xi_i(u_1)^{-1} \otimes 1 + \oint_{C_2} \frac{1}{u_2 - v} \xi_i(v) \otimes \xi_i(u_1) x_j^+(v) \xi_i(u_1)^{-1} dv
\]

\[
= \xi_i(u_1) x_j^+(u_2) \xi_i(u_1)^{-1} \otimes 1 + \oint_{C_2} \frac{u_1 - v + a}{(u_2 - v)(u_1 - v - a)} \xi_i(v) \otimes x_j^+(v) dv
\]

\[
- \oint_{C_2} \frac{2a}{(u_2 - v)(u_1 - v - a)} \xi_i(v) \otimes x_j^+(u_1 - a) dv
\]

where the third summand is equal to zero since the integrand is regular inside \( C_2 \).

Applying \( \Delta_0 \) to the right–hand side yields

\[
\xi_i(u_1) x_j^+(u_2) \xi_i(u_1)^{-1} \otimes 1 + \frac{1}{u_1 - u_2 - a} \oint_{C_2} \left( \frac{u_1 - u_2 + a}{u_2 - v} - \frac{2a}{u_1 - a - v} \right) \xi_i(v) \otimes x_j^+(v) dv
\]

The equality of the two expressions now follows from the identity

\[
\frac{u_1 - u_2 + a}{u_2 - v} - \frac{2a}{u_1 - a - v} = \frac{(u_1 - u_2 - a)(u_1 + a - v)}{(u_2 - v)(u_1 - a - v)}
\]

3.7. **Proof of (Y4).** We check this relation for the + case only. We need to prove that \( \Delta_0 \) preserves the relation

\[
x_{i,r+1} x_{j,s} - x_{i,r} x_{j,s+1} - a x_{i,r} x_{j,s} = x_{j,s} x_{i,r+1} - x_{j,s+1} x_{i,r} + a x_{j,s} x_{i,r}
\]

where \( a = \hbar d_i a_{ij} / 2 \). Note that \( \Delta_0(x_{i,m}x_{j,n}) \) is equal to

\[
x_{i,m}x_{j,n} \otimes 1 + \frac{1}{h} \oint_{C_2} v^m x_{i,m} \xi_j(v) \otimes x_j(v) dv + \frac{1}{h} \oint_{C_2} v^m \xi_i(v)x_{j,n} \otimes x_i(v) dv
\]

\[
+ \frac{1}{h^2} \oint_{C_2} v^m v^2 \xi_i(v_1) \xi_j(v_2) \otimes x_i(v_1)x_j(v_2) dv_1 dv_2
\]
We now apply $\Delta_0$ to both sides of relation (Y4), and consider the four summand of $\Delta_0(x_{i,m}x_{j,n})$ separately.

The first summand of $\Delta_0$ of the left and right–hand sides of (Y4) are, respectively
\[
\left(x_{i,r+1}x_{j,s} - x_{i,r}x_{j,s+1} - ax_{i,r}x_{j,s}\right) \otimes 1
\]
which cancel because of (Y4).

The second summand of the left–hand side and the third summand of the right–hand side are, respectively
\[
\frac{1}{\hbar} \oint_{C_2} v^s(x_{i,r+1} - vx_{i,r} - ax_{i,r})\xi_j(v) \otimes x_j(v) \, dv
\]
which cancel because of the following version of (Y2) and (Y3)
\[
(x_{i,r+1} - vx_{i,r} - ax_{i,r})\xi_j(v) = \xi_j(v)(x_{i,r+1} - vx_{i,r} + ax_{i,r})
\]
Similarly the third summand of the right–hand side and the second summand of the left–hand side cancel.

The fourth summands of the left and right–hand sides of (Y4) are, respectively
\[
\frac{1}{\hbar^2} \oint_{C_2} v_1 v_2^s(v_1 - v_2 - a)\xi_i(v_1)\xi_j(v_2) \otimes x_i(v_1)x_j(v_2) \, dv_1dv_2
\]
\[
\frac{1}{\hbar^2} \oint_{C_2} v_1 v_2^s(v_1 - v_2 + a)\xi_j(v_2)\xi_i(v_1) \otimes x_j(v_2)x_i(v_1) \, dv_1dv_2
\]
By (Y4), their difference is equal to
\[
\frac{1}{\hbar} \oint_{C_2} v_1^s v_2^s \xi_i(v_1)\xi_j(v_2) \otimes ([x_i,0, x_j(v_2)] - [x_i(v_1), x_j,0]) \, dv_1dv_2
\]
which is equal to zero because the first (resp. second) summand is regular when $v_1$ (resp. $v_2$) lies inside $C_2$.

3.8. Proof of (Y5). We need to check that $\Delta_0$ preserves the relation
\[
[x_i^+(u_1), x_j^-(u_2)] = -\hbar \delta_{ij} \frac{\xi_i(u_1) - \xi_i(u_2)}{u_1 - u_2}
\]
Applying $\Delta_0$ to the left–hand side yields
\[
\oint_{C_1} \frac{1}{u_2 - v} [x_i^+(u_1), x_j^-(v)] \otimes \xi_j(v) \, dv
\]
\[
+ \oint_{C_2} \frac{1}{u_1 - v} \xi_i(v) \otimes [x_i^+(v), x_j^-(u_2)] \, dv + B
\]
where

\[ B = \oint_{C_1} \oint_{C_2} \frac{1}{(u_1 - v_2)(u_2 - v_1)} [\xi_i(v_2) \otimes x^+_i(v_2), x^{-}_j(v_1) \otimes \xi_j(v_1)] dv_2 dv_1 \]

We shall prove below that \( B = 0 \). Thus, by relation (Y5) the above is equal to zero if \( i \neq j \). If \( i = j \), it is equal to

\[ - \oint_{C_1} \frac{h}{(v_2 - v)(u_1 - v)} (\xi_i(u_1) - \xi_i(v)) \otimes \xi_i(v) dv \]

\[ - \oint_{C_2} \frac{h}{(u_1 - v)(v - u_2)} \xi_i(v) \otimes (\xi_i(v) - \xi_i(u_2)) dv \]

\[ = \oint_{C_1 \cup C_2} \frac{h}{(u_1 - v)(u_2 - v)} \xi_i(v) \otimes \xi_i(v) dv \]

\[ = \frac{h}{u_1 - u_2} (\xi_i(u_2) \otimes \xi_i(u_2) - \xi_i(u_1) \otimes \xi_i(u_1)) \]

where the first equality follows because \( \xi_i(u_1) \otimes \xi_i(v) \) (resp. \( \xi_i(v) \otimes \xi_i(u_2) \)) is regular when \( v \) is inside \( C_1 \) (resp. \( C_2 \)), and the second by deformation of contours and the fact that \( \xi_i(v) \otimes \xi_i(v) \) is regular outside \( C_1 \cup C_2 \).

Proof that \( B = 0 \). We shall need the following variant of relation (Y3) of Proposition 2.4.

\[(u - v)[\xi_i(u), x_{x}^+(v)] = \pm a \{\xi_i(u), x_{x}^+(v) - x_{x}^+(u)\} \quad (3.1)\]

where \( a = \frac{\hbar d_i a_{ij}}{2} \) and \( \{x, y\} = xy + yx \). The integrand of \( B \) can be simplified in two different ways. First we write

\[ [\xi_i(v_2) \otimes x^+_i(v_2), x^{-}_j(v_1) \otimes \xi_j(v_1)] \]

\[ = [\xi_i(v_2) \otimes x^+_i(v_2) \xi_j(v_1) + x^{-}_j(v_1) \xi_i(v_2) \otimes [x^+_i(v_2), \xi_j(v_1)] \]

Using (3.1), we get

\[ B = \oint_{C_1} \oint_{C_2} \frac{a}{(u_1 - v_2)(u_2 - v_1)(v_1 - v_2)} \]

\[ \left( \{\xi_i(v_2), x^{-}_j(v_1) - x^-_j(v_2)\} \otimes x^+_i(v_2) \xi_j(v_1) \right. \]

\[ \left. - x^{-}_j(v_1) \xi_i(v_2) \otimes \{\xi_j(v_1), x^+_i(v_2) - x^+_i(v_1)\} \right) \ dv_2 dv_1 \]

\[ = \oint_{C_1} \oint_{C_2} \frac{a}{(u_1 - v_2)(u_2 - v_1)(v_1 - v_2)} \left( \{\xi_i(v_2), x^{-}_j(v_1)\} \otimes x^+_i(v_2) \xi_j(v_1) \right. \]

\[ \left. - x^{-}_j(v_1) \xi_i(v_2) \otimes \{\xi_j(v_1), x^+_i(v_2)\} \right) \ dv_2 dv_1 \]

\[ = \oint_{C_1} \oint_{C_2} \frac{a}{(u_1 - v_2)(u_2 - v_1)(v_1 - v_2)} \left( \xi_i(v_2)x^{-}_j(v_1) \otimes x^+_i(v_2) \xi_j(v_1) \right. \]

\[ \left. - x^{-}_j(v_1) \xi_i(v_2) \otimes \xi_j(v_1)x^+_i(v_2) \right) \ dv_2 dv_1 \]
where the second equality follows from the fact that \( \{ \xi_i(v_2), x_j(v_2) \} \otimes x_i^+(v_2) \xi_j(v_1) \)
(resp. \( x_j^-(v_1) \xi_i(v_2) \otimes \{ \xi_j(v_1), x_i^+(v_1) \} \)) is regular when \( v_1 \) is inside \( C_1 \) (resp. \( v_2 \) is inside \( C_2 \)).

Now if we write instead
\[
\{ \xi_i(v_2) \otimes x_i^+(v_2), x_j^-(v_1) \otimes \xi_j(v_1) \} = \{ \xi_i(v_2), x_j^-(v_1) \otimes x_i^+(v_2) \xi_j(v_1) \} + \{ \xi_i(v_2), x_j^-(v_1) \otimes \xi_j(v_1) x_i^+(v_2) \}
\]
and use relation (3.1) as before, we obtain
\[
B = \int_{C_1} \int_{C_2} \frac{-a}{(v_1 - v_2)(u_1 - v_2)(u_2 - v_1)} \left( \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes \xi_j(v_1) x_i^+(v_2) \right) \, dv_2 dv_1
\]
Thus \( B = -B \), whence \( B = 0 \).

3.9. Coassociativity. We need to show that the generators of \( Y_h(\mathfrak{g}) \) act by the same operators on
\[
(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 \quad \text{and} \quad V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)
\]
The action of \( \xi_i(u) \) on both modules is given by \( \xi_i(u - s_1 - s_2) \otimes \xi_i(u - s_2) \otimes \xi_i(u) \).

To compute the action of \( x_i^+(u) \), we shall assume that \( s_1 \) and \( s_2 \) are such that \( \sigma(V_1) + s_1 + s_2, \sigma(V_2) + s_2 \) and \( \sigma(V_3) \) are all disjoint. By (vi), this implies in particular that \( \sigma(V_1 \otimes_{s_1} V_2) + s_2 \) and \( \sigma(V_3) \) are disjoint, and that so are \( \sigma(V_1) + s_1 + s_2 \) and \( \sigma(V_2 \otimes_{s_2} V_3) \), so that the above tensor products are defined.

Under these assumptions, the action of \( x_i^+(u) \) on \( (V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 \) is given by
\[
\Delta_{s_1}(x_i^+(u - s_2)) \otimes 1 + \int_{C_3} \frac{1}{u - v_3} \Delta_{s_1}(\xi_i(v_3 - s_2)) \otimes x_i^+(v_3) \, dv_3
\]
\[
= x_i^+(u - s_2 - s_1) \otimes 1 \otimes 1 + \int_{C_3} \frac{1}{u - s_2 - v_2} \xi_i(v_2 - s_1) \otimes x_i^+(v_2) \otimes 1 \, dv_2
\]
\[
+ \int_{C_3} \frac{1}{u - v_3} \xi_i(v_3 - s_2 - s_1) \otimes \xi_i(v_3 - s_2) \otimes x_i^+(v_3) \, dv_3
\]
where \( C_3 \) encloses \( \sigma(V_3) \) and none of the points of \( \sigma(V_1) + s_1 + s_2 \) and \( \sigma(V_2) + s_2 \). \( C_2 \) encloses \( \sigma(V_2) \) and none of the points of \( \sigma(V_1) + s_1 \), and \( u \) is assumed to be outside of \( C_3 \) and \( C_2 + s_2 \).
The action of $x_i^+(u)$ on $V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)$ is given by

$$x_i^+(u - s_1 - s_2) \otimes 1 \otimes 1 + \int_{C_{23}} \frac{1}{u - v_{23}} \xi_i(v_{23} - s_1 - s_2) \otimes \Delta_{s_2}(x_i^+(v_{23})) \, dv_{23}$$

$$= x_i^+(u - s_1 - s_2) \otimes 1 \otimes 1$$

$$+ \int_{C_{23}} \frac{1}{u - v_{23}} \xi_i(v_{23} - s_1 - s_2) \otimes x_i^+(v_{23} - s_2) \otimes 1 \, dv_{23}$$

$$+ \int_{C_{23}} \int_{C'_3} \frac{1}{u - v_{23} v_{23} - v'_3} \xi_i(v_{23} - s_1 - s_2) \otimes \xi(v'_3 - s_2) \otimes x_i^+(v'_3) \, dv'_3 \, dv_{23}$$

where $C_{23}$ encloses $\sigma(V_2) + s_2 \cup \sigma(V_3)$ and none of the points of $\sigma(V_1) + s_1 + s_2$, $C'_3$ is chosen inside $C_{23}$ and encloses $\sigma(V_3)$ and none of the points of $\sigma(V_2) + s_2$, and $u$ is assumed to be outside of $C_{23}$.

Since the singularities of the first integrand which are contained in $C_{23}$ lie in $\sigma(V_2) + s_2$, the corresponding integral is equal to

$$\int_{C'_{23}} \frac{1}{u - v'_2} \xi_i(v'_2 - s_1 - s_2) \otimes x_i^+(v'_2 - s_2) \otimes 1 \, dv'_2$$

where $C'_{23}$ contains $\sigma(V_2) + s_2$ and none of the points of $\sigma(V_1) + s_1 + s_2$. On the other hand, integrating in $v_{23}$ in the second integral yields

$$\int_{C'_3} \frac{1}{u - v'_3} \xi_i(v'_3 - s_1 - s_2) \otimes \xi(v'_3 - s_2) \otimes x_i^+(v'_3) \, dv'_3$$

so that the two actions of $x_i^+(u)$ agree. The proof for $x_i^-(u)$ is similar.

4. The commutative $R$-matrix of the Yangian

In this section, we construct the commutative part $R^0(s)$ of the $R$-matrix of the Yangian, and show that it defines meromorphic commutativity constraints on $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, when the latter is equipped with the Drinfeld tensor product defined in §3.

A conjectural formula expressing $R^0(s)$ as a formal infinite product with values in the double Yangian $DY_{\hbar}(\mathfrak{g})$ was given by Khoroshkin–Tolstoy [19, Thm. 5.2]. We review this formula in §4.1–4.2, and outline our construction in 4.3. Our starting point is the observation that $R^0(s)$ formally satisfies an additive difference equation whose coefficient matrix $A(s)$ we show to be a rational function on finite–dimensional representations of $Y_{\hbar}(\mathfrak{g})$. By taking the left and right canonical fundamental solutions of this equation, we construct two regularisations $R^{0,\pm}(s)$ of $R^0(s)$ which are meromorphic functions of the parameter $s$, and then show that they have the required intertwining properties with respect to the Drinfeld coproduct.

4.1. The $T$–Cartan matrix of $\mathfrak{g}$. Let $A = (a_{ij})$ be the Cartan matrix of $\mathfrak{g}$ and $B = (b_{ij})$ its symmetrization, where $b_{ij} = d_{ij}a_{ij}$. Let $T$ be an indeterminate, and let $B(T) = ([b_{ij}]T) \in GL_1(\mathbb{C}[T^{\pm1}])$ the corresponding
Then, there exists an integer \( l = m h^\vee \), which is a multiple of the dual Coxeter number \( h^\vee \) of \( g \), and is such that

\[
B(T)^{-1} = \frac{1}{|\mathbb{T}|} C(T)
\]

(4.1)

where the entries of \( C(T) \) are Laurent polynomials in \( T \) with positive integer coefficients.\(^6\) We denote the entries of the matrix \( C(T) \) by \( c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r \), and note that \( c_{ij}(T) = c_{ij}(T^{-1}) \).

4.2. The Khoroshkin–Tolstoy construction. The starting point of [19] is a conjectural presentation of the Drinfeld double \( DY_h(g) \) of the Yangian \( Y_h(g) \). This presentation includes two sets of commuting generators \( \{\xi_{i,r}\}_{i \in \mathbb{I}, r \in \mathbb{Z}_{\geq 0}} \) and \( \{\xi_{i,r}\}_{i \in \mathbb{I}, r \in \mathbb{Z}_{< 0}} \), where the first are the commuting generators of \( Y_h(g) \). Let \( Y_0^\pm \subset DY_h(g) \) be the subalgebras they generate. The Hopf pairing \( \langle -,- \rangle \) on \( DY_h(g) \) restricts to a perfect pairing \( Y_0^+ \otimes Y_0^- \to \mathbb{C} \), and the commutative part of the \( R \)-matrix of \( Y_h(g) \) is given by

\[
\mathcal{R}^0 = \exp \left( \sum_{i \in \mathbb{I}, r \in \mathbb{N}} a_{i,r}^+ \otimes a_{i,-r-1}^- \right)
\]

(4.2)

where \( \{a_{i,r}^+\}_{i \in \mathbb{I}, r \in \mathbb{Z}_{\geq 0}} \) and \( \{a_{i,r}^-\}_{i \in \mathbb{I}, r \in \mathbb{Z}_{< 0}} \) are generators of \( Y_0^+, Y_0^- \) respectively, which are primitive modulo elements which pair trivially with \( Y_0^\pm \), and such that \( \langle a_{i,r}^+, a_{j,-s-1}^- \rangle = \delta_{ij} \delta_{rs} \).

Constructing these generators amounts to finding formal power series

\[
a_i^+(u) = \sum_{r \geq 0} a_{i,r}^+ u^{-r-1} \in Y_0^+[u^{-1}] \quad \text{and} \quad a_i^-(v) = \sum_{r < 0} a_{i,r}^- v^{-r-1} \in Y_0^-[v]
\]

such that \( \langle a_i^+(u), a_j^-(v) \rangle = \delta_{ij} / (u-v) \). To this end, introduce the generating series

\[
\xi_i^+(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad \xi_i^-(v) = 1 - h \sum_{r < 0} \xi_{i,r} v^{-r-1}
\]

Then, the commutation relations of \( DY_h(g) \) imply that

\[
\langle \xi_i^+(u), \xi_j^-(v) \rangle = \frac{u-v+a}{u-v-a} \in \mathbb{C}[u^{-1},v]
\]

where \( a = hh_{ij}/2 \). Define now

\[
t_i^+(u) = \log(\xi_i^+(u)) \in Y_0^+[u^{-1}] \quad \text{and} \quad t_i^-(v) = \log(\xi_i^-(v)) \in Y_0^-[v]
\]

(4.3)

Then, it follows that

\[
\langle t_i^+(u), t_j^-(v) \rangle = \log \left( \frac{u-v+a}{u-v-a} \right)
\]

\(^6\)This result is stated without proof in [19, p. 391], and proved for \( g \) simply–laced in [15, Prop. 2.1]. We give a proof in Appendix A, which also corrects the values of the multiple \( m \) tabulated in [19] for the \( C_n \) and \( D_n \) series. With those corrections, the value of \( m \) for any \( g \) is the ratio of the squared length of long roots and short ones.
Indeed, $\xi^+_i(u)$ are group–like modulo terms which pair trivially with $Y^+_0, Y^-_0$, and if $a, b$ are primitive elements of a Hopf algebra endowed with a Hopf pairing $\langle -,- \rangle$, then $\langle e^a, e^b \rangle = e^{(a,b)}$. Differentiating with respect to $u$ yields

$$\langle \frac{d}{du} t^+_i(u), t^-_j(v) \rangle = \frac{1}{u-v+a} - \frac{1}{u-v-a}$$

Let $T$ be the shift operator acting on functions of $v$ as $T f(v) = f(v-h/2)$. Then, the above identity may be rewritten as

$$\langle \frac{d}{du} t^+_i(u), t^-_j(v) \rangle = (T^{b_{ij}} - T^{-b_{ij}}) \frac{1}{u-v} = (T - T^{-1}) B(T)_{ij} \frac{1}{u-v}$$

where $B(T)$ is the matrix introduced in 4.1. It follows that if $D(T)$ is an $I \times I$ matrix with entries in $C[[T, T^{-1}]]$, then

$$\sum_k D(T)_{jk} \langle \frac{d}{du} t^+_i(u), t^-_k(v) \rangle = (T - T^{-1}) (D(T) B(T))_{ji} \frac{1}{u-v}$$

By (4.1), choosing $D(T) = (T^l - T^{-l})^{-1} C(T)$, and setting

$$a^+_i(u) = \frac{d}{du} t^+_i(u) \quad \text{and} \quad a^-_j(v) = \sum_{k \in I} (T^l - T^{-l})^{-1} C(T)_{jk} t^-_k(v) \quad (4.4)$$

gives the sought for generators, provided one can interpret $(T^l - T^{-l})^{-1}$. This can be done by expanding in powers of $T^l$ or of $T^{-l}$, and leads to two distinct formal expressions for $R^0$ [19, (5.27)–(5.28)].

4.3. To make sense of the above construction of $R^0$ on the tensor product $V_1 \otimes V_2$ of two finite–dimensional representations of $Y_h(\mathfrak{g})$, we proceed as follows.

1. By 2.10,

$$a^+_i(u) = \frac{d}{du} t^+_i(u) = \xi^+_i(u)/\xi^+_i(u)^{-1}$$

is a rational function of $u$, regular near $\infty$.

2. If $a^-_j(v)$ defined by (4.4) can be shown to be a meromorphic function of $v$, we may interpret the sum over $r$ in (4.2) as the contour integral $\oint_C a^+_i(u) \otimes a^-_i(u) du$, where $C$ encloses all poles of $a^+_i(u)$ and none of those of $a^-_i(u)$.

3. The action of $R_0$ on $V_1(s) \otimes V_2$ would then be given by

$$R^0(s) = \exp \left( \sum_i \oint_{C+s} a^+_i(u-s) \otimes a^-_i(u) du \right) = \exp \left( \sum_i \oint_C a^+_i(u) \otimes a^-_i(u+s) du \right)$$

where $C$ encloses all poles of $a^+_i(u)$ on $V_1$ and none of those of $a^-_i(u)$ on $V_2$.

4. We show in 4.4 that, on any finite–dimensional representation of $Y_h(\mathfrak{g})$, $t^+_i(u)$ is the expansion near $u = \infty$ of a meromorphic function of $u$ defined on the complement of a compact cut set $0 \in X \subset \mathbb{C}$, and interpret $t^-_i(v)$ as the corresponding analytic continuation of
t^+_i(u). This resolves in particular the ambiguity in the definition (4.3) of \( t^-_i(v) \) as a formal power series in \( v \), since the constant term of \( \xi^-(v) \) is not equal to 1, and allows to apply the shift operator \( T \) to \( t^-_i(v) \), since \( T \) does not act on formal power series of \( v \).

(5) To interpret \( a^-_j(v) \), we note that if formally satisfies the difference equation

\[
b_j(v) = - \sum_{k \in \mathbf{I}} T^{-1} C(T)_{jk} t^-_k(v) = - \sum_{k \in \mathbf{I}, r \in \mathbb{Z}} c_{jk}^{(r)} t^-_k(v + (l + r)\frac{h}{2})
\]

and we used the fact that \( C(T) = C(T^{-1}) \). This implies that \( R^0(s) \) formally satisfies

\[
R^0(s + l h) R^0(s)^{-1} = \exp\left( \sum_i \int_C a^+_i(u) \otimes b^-_i(u + s) \right) \tag{4.5}
\]

(6) We show in 4.5–4.7 that the operator \( A(s) \) given by the right-hand side of (4.5) is a rational function of \( s \) such that \( A(\infty) = 1 \). We then define two regularisations \( R^{0,\pm}(s) \) of \( R^0(s) \) as the canonical right and left fundamental solutions of the difference equation (4.5), and show in 4.9 that they define meromorphic commutativity constraints on \( \text{Rep}_{fd}(Y_h(g)) \) endowed with the deformed Drinfeld coproduct.

4.4. Matrix logarithms. We shall need the following result

**Proposition.** Let \( V \) be a complex, finite-dimensional vector space, and \( \xi : \mathbb{C} \to \text{End}(V) \) a rational function such that

- \( \xi(\infty) = 1 \).
- \( [\xi(u), \xi(v)] = 0 \) for any \( u, v \in \mathbb{C} \).

Let \( \sigma(\xi) \subset \mathbb{C} \) be the set of poles of \( \xi(u)^{\pm 1} \), and define the cut-set \( X(\xi) \) by

\[
X(\xi) = \bigcup_{a \in \sigma(\xi)} [0, a] \tag{4.6}
\]

where \( [0, a] \) is the line segment joining 0 and \( a \). Then, there is a unique single-valued, holomorphic function \( t(u) = \log(\xi(u)) : \mathbb{C} \setminus X(\xi) \to \text{End}(V) \) such that

\[
\exp(t(u)) = \xi(u) \quad \text{and} \quad t(\infty) = 0 \tag{4.7}
\]

Moreover, \( [t(u), t(v)] = 0 \) for any \( u, v \in \mathbb{C} \), and \( t(u)' = \xi(u)^{-1}\xi'(u) \).

**Proof.** The equation (4.7) uniquely defines \( t(u) \) as a holomorphic function near \( u = \infty \). To continue \( t(u) \) meromorphically, note first that the semisimple and unipotent factors \( \xi_S(u), \xi_U(u) \) of the multiplicative Jordan
decomposition of $\xi(u)$ are rational functions of $u$ since $[\xi(u),\xi(v)] = 0$ for any $u,v$ (see e.g., [11, Lemma 4.12]). Thus,

$$t_N(u) = \log(\xi_U(u)) = \sum_{k \geq 1} (-1)^{k-1} \frac{(\xi_U(u) - 1)^k}{k}$$

is a well–defined rational function of $u \in \mathbb{C}$ whose poles are contained in those of $\xi(u)$.

To define $\log(\xi_S(u))$ consistently, note that the eigenvalues of $\xi(u)$ are rational functions of the form $\prod_{j}(u - a_j)(u - b_j)^{-1}$. Since, for $a \in \mathbb{C}^\times$, the function $\log(1 - au^{-1})$ is single–valued on the complement of the interval $[0,a]$, where $\log$ is the standard determination of the logarithm, we may define a single–valued, holomorphic function $\log(\xi_S(u))$ on the complement of the intervals $[0,a]$, where $a$ ranges over the (non–zero) zeros and poles of the eigenvalues of $\xi(u)$.

Finally, we set

$$t(u) = t_N(u) + t_S(u)$$

The fact that $[t(u),t(v)] = 0$ is clear from the construction, or from the fact that it clearly holds for $u,v$ near $\infty$. Finally, the derivative of $t(u)$ can be computed by differentiating the identity $\exp(t(u)) = \xi(u)$, and using the formula for the left–logarithmic derivative of the exponential function (see, e.g., [9]).

**Definition.** If $V$ is a finite–dimensional representation of $\mathfrak{y}_h(\mathfrak{g})$, and $\xi_i(u)$ is the rational function $\xi_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1}$ given by Proposition 2.10, the corresponding logarithm will be denoted by $t_i(u)$.

**Theorem.** The operator $\mathcal{A}_{V_1,V_2}(s)$. Let $V_1,V_2$ be two finite–dimensional representations of $\mathfrak{y}_h(\mathfrak{g})$. Let $\mathcal{C}_1$ be a contour enclosing the set of poles $\sigma(V_1)$ of $V_1$, and consider the following operator on $V_1 \otimes V_2$

$$\mathcal{A}_{V_1,V_2}(s) = \exp \left( - \sum_{i,j \in \mathbb{I}, r \in \mathbb{Z}} c_{ij}^{(r)} \oint_{\mathcal{C}_1} t'_j(v) \otimes t_j \left( v + s + \frac{(l+r)h}{2} \right) \, dv \right)$$

where $s \in \mathbb{C}$ is such that $t_j(v + s + h(l+r)/2)$ is an analytic function of $v$ within $\mathcal{C}_1$ for every $j \in \mathbb{I}$ and $r \in \mathbb{Z}$ such that $c_{ij}^{(r)} \neq 0$ for some $i \in \mathbb{I}$.

**Theorem.**

(i) $\mathcal{A}_{V_1,V_2}(s)$ extends to a rational function of $s$ which is regular at $\infty$, and such that

$$\mathcal{A}_{V_1,V_2}(s) = 1 - \frac{\hbar^2 \Omega_h}{s^2} + O(s^{-3})$$

where $\Omega_h = \sum_i d_i h_i \otimes \varpi_i^y \in \mathfrak{h} \otimes \mathfrak{h}$. The poles of $\mathcal{A}_{V_1,V_2}(s)^{\pm 1}$ are contained in

$$\sigma(V_2) - \sigma(V_1) - \frac{\hbar}{2} \{l + r\}$$
where \( r \) ranges over the integers such that \( c_{ij}^{(r)} \neq 0 \) for some \( i, j \in I \).

(ii) For any \( s, s' \) we have \([\mathcal{A}_{V_1, V_2}(s), \mathcal{A}_{V_1, V_2}(s')] = 0\).

(iii) For any \( V_1, V_2, V_3 \in \text{Rep}_d(Y_h(g)) \), we have
\[
\mathcal{A}_{V_1 \otimes_{a_1} V_2, V_3}(s_2) = \mathcal{A}_{V_1, V_3}(s_1 + s_2) \mathcal{A}_{V_2, V_3}(s_2)
\]
\[
\mathcal{A}_{V_1, V_2 \otimes_{a_2} V_3}(s_1 + s_2) = \mathcal{A}_{V_1, V_3}(s_1 + s_2) \mathcal{A}_{V_1, V_2}(s_1)
\]

(iv) The following shifted unitary condition holds
\[
\sigma \circ \mathcal{A}_{V_1, V_2}(-s) \circ \sigma^{-1} = \mathcal{A}_{V_2, V_1}(s - l\hbar)
\]
where \( \sigma : V_1 \otimes V_2 \to V_2 \otimes V_1 \) is the flip of the tensor factors.

(v) For every \( a, b \in \mathbb{C} \) we have
\[
\mathcal{A}_{V_1(a), V_2(b)}(s) = \mathcal{A}_{V_1, V_2}(s + a - b)
\]

**Proof.** Properties (ii), (iii) and (v) follow from the definition of \( \mathcal{A} \), and the fact that \( t_i(u) \) are primitive with respect to the Drinfeld coproduct. To prove (i) and (iv), we work in the following more general situation.

Let \( V, W \) be complex, finite–dimensional vector spaces, \( A, B : \mathbb{C} \to \text{End}(V) \) rational functions satisfying the assumptions of Proposition 4.4, and let \( \log A(v), \log B(v) \) be the corresponding logarithms. Let \( \sigma(A), \sigma(B) \) denote the set of poles of \( A(v)^{\pm 1} \) and \( B(v)^{\pm 1} \) respectively. Set
\[
X(s) = \exp \left( \oint_{C_1} A(v)^{-1} A'(v) \otimes \log(B(v + s)) \, dv \right)
\]
where \( C_1 \) encloses \( \sigma(A) \), and \( s \) is such that \( \log(B(v + s)) \) is analytic within \( C_1 \).

**Claim 1.** The operator \( X(s) \in \text{End}(V \otimes W) \) is a rational function of \( s \), regular at \( \infty \), and has the following Taylor series expansion near \( \infty \)
\[
X(s) = 1 + (A_0 \otimes B_0)s^{-2} + O(s^{-3})
\]
where \( A(s) = 1 + A_0 s^{-1} + O(s^{-2}) \) and \( B(s) = 1 + B_0 s^{-1} + O(s^{-2}) \). Moreover, the poles of \( X(s)^{\pm 1} \) are contained in \( \sigma(B) - \sigma(A) \).

Note that this claim implies the first part of Theorem 4.5, since
\[
\mathcal{A}_{V_1, V_2}(s) = \prod_{i,j \in I} \exp \left( \oint_{C} t'_i(v) \otimes t_j \left( v + s + \frac{(l + r)\hbar}{2} \right) \, dv \right) c_{ij}^{(r)}
\]
\[
= 1 - h^2 s^{-2} \sum_{i,j \in I} c_{ij}^{(r)} \xi_{i,0} \otimes \xi_{j,0} + O(s^{-3})
\]
\[
= 1 - l h^2 \Omega_b s^{-2} + O(s^{-3})
\]
since \( c_{ij}(T)|_{T=1} \) is the \((i, j)\) entry of \( l \cdot B^{-1} \).

Part (iv) of Theorem 4.5 is a consequence of the following claim, together with the fact that \( c_{ij}^{(r)} = c_{ij}^{(r)} = c_{ij}^{(-r)} \).
Claim 2. $X(s) = \exp \left( \oint_{C_2} \log(A(v - s)) \otimes B(v)^{-1} B'(v) \, dv \right)$, where $C_2$ encloses $\sigma(B)$ and $s \in \mathbb{C}$ is such that $\log(A(v - s))$ is analytic within $C_2$.

We prove these claims in §4.6 and 4.7 respectively. □

4.6. Proof of Claim 1. Since $A(v)$ commutes with itself for different values of $v$, the semisimple and unipotent parts $A(v) = A_S(v)A_U(v)$ of the Jordan decomposition of $A(v)$ are rational functions of $v$ [11, Lemma 4.12]. Since the logarithmic derivative of $A(v)$ separates the two additively, we can treat the semisimple and unipotent cases separately.

The semisimple case reduces to the scalar case, i.e., when $V$ is one–dimensional and

$$A(v) = \prod_j \frac{v - a_j}{v - b_j} = 1 + (\sum_j b_j - a_j)v^{-1} + O(v^{-2})$$

for some $a_j, b_j \in \mathbb{C}$. In this case,

$$X(s) = \exp \left( \sum_j \oint_{C_1} \left( \frac{1}{v - a_j} - \frac{1}{v - b_j} \right) \otimes \log(B(v + s)) \, dv \right)$$

$$= \exp \left( \sum_j 1 \otimes (\log(B(s + a_j)) - \log(B(s + b_j))) \right)$$

$$= \prod_j 1 \otimes B(s + a_j)B(s + b_j)^{-1}$$

which is a rational function of $s$ such that the poles of $X(s)^{\pm 1}$ are contained in $\sigma(B) - \sigma(A)$. Moreover,

$$X(s) = 1 + s^{-2} \left( \sum_j b_j - a_j \right) \otimes B_0 + O(s^{-3})$$

Assume now that $A(v)$ is unipotent. In this case,

$$\log(A(v)) = \sum_{k \geq 1} (-1)^{k-1} \frac{(A(v) - 1)^k}{k} = A_0 v^{-1} + O(v^{-2})$$

is given by a finite sum, and is therefore a rational function of $v$. Decomposing it into partial fractions yields

$$\log(A(v)) = \sum_{j \in I} \sum_{n \in \mathbb{N}} \frac{N_{j,n}}{(v - a_j)^{n+1}}$$
where $J$ is a finite indexing set, $a_j \in \mathbb{C}$ and $\sum_j N_{j,0} = A_0$. In this case we obtain

$$X(s) = \exp \left( \sum_{j \in J} -(n+1)N_{j,n} \otimes \frac{\partial^{n+1}}{(n+1)!} \log(B(v)) \bigg|_{v=s+a_j} \right)$$

This is again a rational function of $s$ since the $N_{j,n}$ are nilpotent and pairwise commute, such that the poles of $X(s)^{\pm1}$ are contained in $\sigma(B) - \sigma(A)$. Moreover,

$$X(s) = 1 + s^{-2} \sum_j N_{j,0} \otimes B_0 + O(s^{-3})$$

4.7. **Proof of Claim 2.** Let $X(A), X(B) \subset \mathbb{C}$ be defined by (4.6), and $C_1, C_2$ be two contours enclosing $X(A)$ and $X(B)$ respectively. For each $s \in \mathbb{C}$ such that $C_1 + s$ is outside of $C_2$, we have

$$\oint_{C_1} A(v)^{-1} A'(v) \otimes \log(B(v+s)) \, dv$$

$$= -\oint_{C_1} \log(A(v)) \otimes B(v+s)^{-1} B'(v+s) \, dv$$

$$= \oint_{C_2-s} \log(A(v)) \otimes B(v+s)^{-1} B'(v+s) \, dv$$

$$= \oint_{C_2} \log(A(w-s)) \otimes B(w)^{-1} B'(w) \, dv$$

where the first equality follows by integration by parts, the second by a deformation of contour since the integrand is regular at $v = \infty$ and has zero residue there, and the third by the change of variables $w = v + s$.

4.8. **The abelian $R$–matrix of $Y_b(g)$.** Let $V_1, V_2 \in \text{Rep}_{qN}(Y_b(g))$, and let $\mathcal{A}_{V_1,V_2}(s) \in GL(V_1 \otimes V_2)$ be the operator defined in 4.5. Consider the additive difference equation

$$\mathcal{R}_{V_1,V_2}(s + lh) = \mathcal{A}_{V_1,V_2}(s) \mathcal{R}_{V_1,V_2}(s)$$

where $l \in \mathbb{N}$ is given by (4.1).

This equation is regular, in that $\mathcal{A}_{V_1,V_2}(s) = 1 + O(s^{-2})$ by Theorem 4.5. In particular, it admits two canonical meromorphic fundamental solutions

$$\mathcal{R}_{V_1,V_2}^{0,\pm} : \mathbb{C} \to GL(V_1 \otimes V_2)$$

which are uniquely determined by the following requirements (see e.g., [2, 3, 20] or [11, §4])

- $\mathcal{R}_{V_1,V_2}^{0,\pm}(s)$ is holomorphic and invertible for $\pm \text{Re}(s/h) \gg 0$.
- $\mathcal{R}_{V_1,V_2}^{0,\pm}(s)$ possesses an asymptotic expansion of the form

$$\mathcal{R}_{V_1,V_2}^{0,\pm}(s) \sim 1 + R_0^\pm s^{-1} + R_1^\pm s^{-2} + \ldots$$
in any half-plane $\pm \text{Re}(s/h) > m$, $m \in \mathbb{R}$.

Explicitly,

$$\mathcal{R}_{V_1, V_2}^{0, \pm}(s) = \prod_{n \geq 0} \mathcal{A}_{V_1, V_2}(s + nlh)^{-1}$$

$$\mathcal{R}_{V_1, V_2}^{0, -}(s) = \prod_{n \geq 1} \mathcal{A}_{V_1, V_2}(s - nlh)$$

4.9. The following is the main result of this section.

**Theorem.** $\mathcal{R}_{V_1, V_2}^{0, \pm}(s)$ have the following properties

(i) The map

$$\sigma \circ \mathcal{R}_{V_1, V_2}^{0, \pm}(s) : V_1(s) \otimes_0 V_2 \to V_2 \otimes_0 V_1(s)$$

where $\sigma$ is the flip of tensor factors, is a morphism of $Y_h(\mathfrak{g})$–modules, which is natural in $V_1$ and $V_2$.

(ii) For any $V_1, V_2, V_3 \in \text{Rep}_{id}(Y_h(\mathfrak{g}))$ we have

$$\mathcal{R}_{V_1, V_2}^{0, \pm}(s_2) = \mathcal{R}_{V_1, V_2}^{0, \pm}(s_1 + s_2) \mathcal{R}_{V_2, V_3}^{0, \pm}(s_2)$$

$$\mathcal{R}_{V_1, V_2 \otimes_2 V_3}^{0, \pm}(s_1 + s_2) = \mathcal{R}_{V_1, V_2}^{0, \pm}(s_1 + s_2) \mathcal{R}_{V_1, V_2}^{0, \pm}(s_1)$$

(iii) The following unitary condition holds

$$\sigma \circ \mathcal{R}_{V_1, V_2}^{0, \pm}(-s) \circ \sigma^{-1} = \mathcal{R}_{V_2, V_1}^{0, \mp}(s)^{-1}$$

(iv) For $a, b \in \mathbb{C}$ we have

$$\mathcal{R}_{V_1(a), V_2(b)}^{0, \pm}(s) = \mathcal{R}_{V_1, V_2}^{0, \pm}(s + a - b)$$

(v) For any $s, s'$,

$$[\mathcal{R}_{V_1, V_2}^{0, \pm}(s), \mathcal{R}_{V_1, V_2}^{0, \pm}(s')] = 0 = [\mathcal{R}_{V_1, V_2}^{0, \pm}(s), \mathcal{R}_{V_1, V_2}^{0, \mp}(s')]$$

(vi) $\mathcal{R}_{V_1, V_2}^{0, \pm}(s)$ have the same asymptotic expansion, with 1–jet

$$\mathcal{R}_{V_1, V_2}^{0, \pm}(s) \sim 1 + h\Omega_h s^{-1} + O(s^{-2}) \quad (4.9)$$

(vii) There is a $\rho > 0$ such that the asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \pm}(s)$ is valid on any domain

$$\{ \pm \text{Re}(s/h) > m \} \cup \{ |\text{Im}(s/h)| > \rho, \arg(\pm s/h) \in (\pi - \delta, \pi + \delta) \}$$

where $m \in \mathbb{R}$ and $\delta \in (0, \pi)$ are arbitrary.

(viii) The poles of $\mathcal{R}_{V_1, V_2}^{0, \pm}(s)^{\pm 1}$ and $\mathcal{R}_{V_1, V_2}^{0, -}(s)^{\pm 1}$ are contained in

$$\sigma(V_2) - \sigma(V_1) - Z_{\geq 0}lh - \frac{h}{2} \{ l + r \} \quad \text{and} \quad \sigma(V_2) - \sigma(V_1) + Z_{> 0}lh - \frac{h}{2} \{ l + r \}$$

where $r$ ranges over the integers such that $c_{ij}^{(r)} \neq 0$ for some $i, j \in \mathbf{I}$. 
PROOF. Part (i) is proved in 4.12 after some preparatory results. Properties (ii)–(vi) and (viii) follow from Theorem 4.5. (vii) is proved in [27, Lemma 8.1].

4.10. Commutation relations with $\mathcal{A}_{V_1,V_2}(s)$. Let $\mathcal{C} \subset \mathbb{C}$ be a contour, and $a_\ell : \mathbb{C} \to \text{End}(V_\ell)$, $\ell = 1, 2$ two meromorphic functions which are analytic within $\mathcal{C}$ and commute with the operators $\{\xi_{i,r}\}_{i \in I,r \in \mathbb{N}}$. For any $k \in I$, define operators $X_k^{\pm,\ell} \in \text{End}(V_1 \otimes V_2)$ by

$$X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v) \otimes a_2(v) \, dv \quad \text{and} \quad X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(v) \otimes a_2(v)x_k^{\pm}(v) \, dv$$

Proposition. The following commutation relations hold

$$\text{Ad}(\mathcal{A}_{V_1,V_2}(s))X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v) \otimes a_2(v)\xi_k(v + s + \ell \hbar)^{\pm 1}\xi_k(v + s)\, dv$$

$$\text{Ad}(\mathcal{A}_{V_1,V_2}(s))X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(v)\xi_k(v - s)^{\pm 1}\xi_k(v - s - \ell \hbar)^{\pm 1} \otimes a_2(v)x_k^{\pm}(v) \, dv$$

PROOF. We only prove the first relation. The second one follows from the first and the unitarity property of Theorem 4.5. We begin by computing the commutation between $X_k^{\pm,1}$ and a typical summand in log $\mathcal{A}_{V_1,V_2}(s)$. Set $b = \pm \hbar d_{a_{ik}}/2$. By (2.1),

$$[\oint_{\mathcal{C}_1} t_i'(u) \otimes t_j(u + s) \, du, X_k^{\pm,1}]$$

$$= \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} a_1(v)[t_i'(u), x_k^{\pm}(v)] \otimes t_j(u + s)a_2(v) \, dvdu$$

$$= \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} \frac{1}{u - v + b} a_1(v)x_k^{\pm}(v) \otimes t_j(u + s)a_2(v) \, dvdu$$

$$- \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} \frac{1}{u - v - b} a_1(v)x_k^{\pm}(v) \otimes t_j(u + s)a_2(v) \, dvdu$$

$$+ \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} \frac{1}{u - v - b} a_1(v)x_k^{\pm}(u - b) \otimes t_j(u + s)a_2(v) \, dvdu$$

$$- \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} \frac{1}{u - v + b} a_1(v)x_k^{\pm}(u + b) \otimes t_j(u + s)a_2(v) \, dvdu$$

$$= \oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v) \otimes (t_j(v - b + s) - t_j(v + b + s))a_2(v) \, dv$$

where the third identity follows by choosing the contour $\mathcal{C}_1$ so that it encloses $\mathcal{C}$ and its translates by $\pm b$, and by using the fact that $s$ is such that $t_j(u + s)$ is holomorphic inside $\mathcal{C}_1$. 

Let the indeterminate $T$ of Section 4.1 act as the difference operator $T_t^j(v) = t^j(v - \hbar/2)$. Then,

$$\sum_{i,j \in I} \int_{C_1} \left[ \oint t^j_t(u) \otimes c_{ij}(T) t_j(u + s) \, du, X^\pm_{k} \right]$$

$$= \sum_{i,j \in I} \int_{C} a_1(v) x^\pm_k(v) \otimes a_2(v) c_{ij}(T)(T^\pm b_k - T^\mp b_k - T^\pm b_k) t_j(v + s) \, dv$$

$$= \pm \int_{C} a_1(v) x^\pm_k(v) \otimes a_2(v)(T^l - T^{-l}) t_j(v + s) \, dv$$

where the second equality follows from (4.1). The claimed identity easily follows from this. □

4.11. Let $X^\pm_{k,1}, X^\pm_{k,2}$ be the operators defined in 4.10. The following is a corollary of Proposition 4.10 and the definition of $R^0_{V_1, V_2}(s)$.

**Proposition.** The following commutation relations hold for any $\varepsilon \in \{\pm\}$

$$\text{Ad}(R^0_{V_1, V_2}(s)) X^\pm_{k,1} = \int_{C} a_1(v) x^\pm_k(v) \otimes a_2(v) \xi_k(v + s)^\pm \, dv$$

$$\text{Ad}(R^0_{V_1, V_2}(s)) X^\pm_{k,2} = \int_{C} a_1(v) \xi_k(v - s)^\mp \otimes a_2(v) x^\pm_k(v) \, dv$$

4.12. **Proof of (i) of Theorem 4.9.** We first rewrite the Drinfeld coproduct in a more symmetric way. Let $V$ be a finite–dimensional representation of $Y_\hbar(g)$ and $C^\pm \subset C$ a contour containing the poles of $x^\pm_i(u)$ on $V$. Then, a simple contour deformation shows that, for any $u$ not contained inside $C^\pm$,

$$\oint_{C^\pm} x^\pm_i(v) \, dv \frac{dv}{u - v} = x^\pm_i(u)$$

It follows that

$$\Delta_s(x^+_i(u)) = \oint_{C_1} x^+_i(v - s) \otimes \frac{dv}{u - v} + \oint_{C_2} \xi_i(v - s) \otimes x^+_i(v) \, dv \frac{dv}{u - v}$$

$$\Delta_s(x^-_i(u)) = \oint_{C_1} x^-_i(v - s) \otimes \xi_i(v) \, dv \frac{dv}{u - v} + \oint_{C_2} 1 \otimes x^-_i(v) \, dv \frac{dv}{u - v}$$

where $C_1, C_2$ are as in 3.4. The result now follows from Proposition 4.11.

5. **The functor $\Gamma$**

We review below the main construction of [11]. Assume henceforth that $h \in \mathbb{C} \setminus \mathbb{Q}$, and that $q = e^{\pi i h}$.
5.1. **Difference equations.** Consider the abelian, additive difference equations
\[ \phi_i(u + 1) = \xi_i(u)\phi_i(u) \]  
(5.1)
defined by the commuting fields \( \xi_i(u) = 1 + h\xi_{i,0}u^{-1} + \cdots \) on a finite-dimensional representation \( V \) of \( Y_h(\mathfrak{g}) \).

Let \( \phi_i^+(u) : \mathbb{C} \to GL(V) \) be the canonical fundamental solutions of (5.1). \( \phi_i^+(u) \) are uniquely determined by the requirement that they be holomorphic and invertible for \( \pm \text{Re}(u) \gg 0 \), and admit an asymptotic expansion of the form
\[ \phi_i^+(u) \sim (1 + \varphi_0^+u^{-1} + \varphi_1^+u^{-2} + \cdots)(\pm u)^h\xi_{i,0} \]
in any right (resp. left) halfplane \( \pm \text{Re}(s) > m \), \( m \in \mathbb{R} \) (see e.g., [2, 3, 20] or [11, §4]). \( \phi_i^+(u) \), \( \phi_i^-(u) \) are regularisations of the formal infinite products
\[ \xi_i(u)^{-1}\xi_i(u + 1)^{-1}\xi_i(u + 2)^{-1}\cdots \quad \text{and} \quad \xi_i(u - 1)\xi_i(u - 2)\xi_i(u - 3)\cdots \]
respectively.

Let \( S_i(u) = (\phi_i^+(u))^{-1}\phi_i^-(u) \) be the connection matrix of (5.1). Thus, \( S_i(u) \) is 1–periodic in \( u \), and therefore a function of \( z = \exp(2\pi iu) \). It is moreover regular at \( z = 0, \infty \), and therefore a rational function of \( z \) such that
\[ S_i(0) = e^{\pi ih\xi_{i,0}} = S_i(\infty)^{-1} \]
Explicitly,
\[ S_i(u) = \lim_{n \to \infty} \xi_i(u + n)\cdots\xi_i(u + 1)\xi_i(u)\xi_i(u - 1)\cdots\xi_i(u - n) \]

5.2. **Non–congruent representations.** We shall say that \( V \in \text{Rep}_{\text{id}}(Y_h(\mathfrak{g})) \) is non–congruent if, for any \( i \in \mathcal{I} \), the poles of \( x_i^+(u) \) (resp. \( x_i^-(u) \)) are not congruent modulo \( \mathbb{Z} \). Let \( \text{Rep}_{\text{id}}^{\text{NC}}(Y_h(\mathfrak{g})) \) be the full subcategory of \( \text{Rep}_{\text{id}}(Y_h(\mathfrak{g})) \) consisting of non–congruent representations.

5.3. **The functor \( \Gamma \).** Given \( V \in \text{Rep}_{\text{id}}^{\text{NC}}(Y_h(\mathfrak{g})) \), define the action of the generators of \( U_q(L\mathfrak{g}) \) on \( \Gamma(V) = V \) as follows.

(i) For any \( i \in \mathcal{I} \), the generating series \( \Psi_i(z)^+ \) (resp. \( \Psi_i(z)^- \)) of the commuting generators of \( U_q(L\mathfrak{g}) \) acts as the Taylor expansions at \( z = \infty \) (resp. \( z = 0 \)) of the rational function
\[ \Psi_i(z) = S_i(u)|_{e^{2\pi iu} = z} \]
To define the action of the remaining generators of \( U_q(L\mathfrak{g}) \), let \( g_i^+(u) : \mathbb{C} \to GL(V) \) be given by \( g_i^+(u) = \phi_i^+(u + 1)^{-1} \) and \( g_i^-(u) = \phi_i^-(u) \). Explicitly,
\[ g_i^+(u) = e^{\pm \gamma h\xi_{i,0}} \prod_{n \geq 1} \xi_i(u \pm n) e^{\pm h\xi_{i,0}/n} \]  
(5.2)
where \( \gamma = \lim_{n \to \infty} (1 + \cdots + 1/n - \log n) \) is the Euler–Mascheroni constant, are regularisations of the infinite products
\[ \xi_i(u + 1)\xi_i(u + 2)\cdots \quad \text{and} \quad \xi_i(u - 1)\xi_i(u - 2)\cdots \]
Note also that, by definition of $g_i^\pm(u)$
\[ S_i(u) = g_i^+(u) \cdot \xi_i(u) \cdot g_i^-(u) \] (5.3)
Let $c_i^\pm \in \mathbb{C}^\times$ be scalars such that $c_i^- c_i^+ = d_i \Gamma(hd_i)^2$.

(ii) For any $i \in I$ and $k \in \mathbb{Z}$, $\mathcal{X}_{i,k}^\pm$ acts as the operator
\[ \mathcal{X}_{i,k}^\pm = c_i^\pm \int_{C_i^\pm} e^{2\pi i k u} g_i^\pm(u) x_i^\pm(u) \, du \]
where the Jordan curve $C_i^\pm$ encloses the poles of $x_i^\pm(u)$ and none of their $\mathbb{Z}^\times$–translates. \(^7\) The corresponding generating series are the expansions at $z = \infty, 0$ of the End($V$)–valued rational function given by
\[ \mathcal{X}_i^\pm(z) = c_i^\pm \int_{C_i^\pm} \frac{z}{z - e^{2\pi i k u}} g_i^\pm(u) x_i^\pm(u) \, du \]
where $z$ lies outside of $\exp(2\pi i C_i^\pm)$.

5.4. Let $\Pi \subset \mathbb{C}$ be a subset such that $\Pi \pm \frac{1}{2} \subset \Pi$. Let
\[ \text{Rep}_{\Pi} \subset \text{Rep}_{\Pi}(Lg) \]
be the full subcategory of consisting of the representations $V$ such that $\sigma(V) \subset \Pi$.

Similarly, let $\Omega \subset \mathbb{C}^\times$ be a subset stable under multiplication by $q^{\pm 1}$. We define $\text{Rep}_{\Omega}(U_q(Lg))$ to be the full subcategory of $\text{Rep}_{\Omega}(U_q(Lg))$ consisting of those $V$ such that $\sigma(V) \subset \Omega$.

5.5.

**Theorem.**

(i) The above operators give rise to an action of $U_q(Lg)$ on $V$. They therefore define an exact, faithful functor
\[ \Gamma : \text{Rep}_{\Pi}^{NC}(Y_h(g)) \rightarrow \text{Rep}_{\Omega}(U_q(Lg)) \]
(ii) The functor $\Gamma$ is compatible with shift automorphisms. That is, for any $V \in \text{Rep}_{\Pi}^{NC}(Y_h(g))$ and $a \in \mathbb{C}$,
\[ \Gamma(V(a)) = \Gamma(V)(e^{2\pi i a}) \]
(iii) Let $\Pi \subset \mathbb{C}$ be a non–congruent subset such that $\Pi \pm \frac{1}{2} \subset \Pi$. Then, $\text{Rep}_{\Pi}^{\Omega}(Y_h(g))$ is a subcategory of $\text{Rep}_{\Pi}^{NC}(Y_h(g))$, and $\Gamma$ restricts to an isomorphism of abelian categories.
\[ \Gamma_{\Pi} : \text{Rep}_{\Pi}^{\Omega}(Y_h(g)) \rightarrow \text{Rep}_{\Pi}^{\Omega}(U_q(Lg)) \]
where $\Omega = \exp(2\pi i \Pi)$.

\(^7\)Note that such a curve exists for any $i \in I$ since $V$ is non–congruent.
6. Tensor structure on $\Gamma$

6.1. The abelian qKZ equations. Let $V_1, V_2$ be finite–dimensional representations of $\mathfrak{g}(\mathfrak{g})$, choose $\varepsilon \in \{\pm\}$, and let $\mathcal{R}_{V_1,V_2}^{0,\varepsilon}(s)$ be the corresponding $R$–matrix defined in 4.9. Consider the abelian, additive $q$KZ equation

$$f(s+1) = \mathcal{R}_{V_1,V_2}^{0,\varepsilon}(s)f(s)$$

(6.1)

Note that this equation does not fit the usual assumptions in the study of difference equations since $\mathcal{R}_{V_1,V_2}^{0,\varepsilon}(s)$ is not rational. Moreover, $\mathcal{R}_{V_1,V_2}^{0,\varepsilon}(s)$ does not have a Laurent expansion at $\infty$, but only an asymptotic expansion of the form $1 + \hbar \Omega/s + O(s^{-2})$ valid in any domain of the form

$$\{\text{Re}(s/\varepsilon h) > m\} \cup \{\text{Im}(s/\varepsilon h) > \rho, \arg(s/\varepsilon h) \in (\pi - \delta, \pi + \delta)\}$$

where $\rho > 0$ is fixed, and $m \in \mathbb{R}$, $\delta \in (0, \pi)$ are arbitrary (see Theorem 4.9).\(^8\)

Nevertheless, these asymptotics and the fact that the poles of $\mathcal{R}_{V_1,V_2}^{0,\varepsilon}(s)\pm 1$ are contained in the complement of a domain of the above form, are sufficient to carry over the standard proofs (see, e.g., [11, §4]) and yield the following.

**Proposition.** Let $n \in \mathbb{C}^\times$ be perpendicular to $h$ and such that $\text{Re}(n) \geq 0$.

(i) If $\varepsilon h \notin \mathbb{R}_{<0}$, the equation (6.1) admits a canonical right meromorphic solution $\Phi_+^\varepsilon : \mathbb{C} \rightarrow \text{GL}(V_1 \otimes V_2)$, which is uniquely determined by the following requirements

- $\Phi_+^\varepsilon$ is holomorphic and invertible for $\text{Re}(s) \gg 0$ if $\text{Re}(\varepsilon h) \geq 0$, and otherwise on a sector of the form

  $$\text{Re}(s) \gg 0 \quad \text{and} \quad \frac{\text{Re}(s)}{\text{Re}(s/n)} \gg 0$$

  (6.2)

- $\Phi_+^\varepsilon$ has an asymptotic expansion of the form $(1 + O(s^{-1}))s^{M_h}$ in any right half–plane if $\text{Re}(\varepsilon h) > 0$, and otherwise in a sector of the form (6.2).

(ii) If $\varepsilon h \notin \mathbb{R}_{>0}$, the equation (6.1) admits a canonical left meromorphic solution $\Phi_-^\varepsilon : \mathbb{C} \rightarrow \text{GL}(V_1 \otimes V_2)$, which is uniquely determined by the following requirements

- $\Phi_-^\varepsilon$ is holomorphic and invertible for $\text{Re}(s) \ll 0$ if $\text{Re}(\varepsilon h) \leq 0$, and otherwise on a sector of the form

  $$\text{Re}(s) \ll 0 \quad \text{and} \quad \frac{\text{Re}(s)}{\text{Re}(s/n)} \ll 0$$

  (6.3)

---

\(^8\)For the qKZ equations determined by the full $R$–matrix, these issues are usually addressed by proving the existence of factorisation $R_{V_1,V_2}(s) = R_{V_1,V_2}^{\text{rat}}(s) \cdot R_{V_1,V_2}^{\text{mer}}(s)$, where $R_{V_1,V_2}^{\text{rat}}(s)$ is a rational function of $s$ which intertwines the Kac–Moody coproduct $\Delta$ and its opposite, and the meromorphic factor $R_{V_1,V_2}^{\text{mer}}(s)$ intertwines $\Delta$ (see [18] for the case of $U_q(\mathfrak{g})$), and then working with $R_{V_1,V_2}^{\text{rat}}(s)$ instead of $R_{V_1,V_2}(s)$. A similar factorisation can be obtained for the abelian $R$–matrices $\mathcal{R}_{V_1,V_2}^{0,\pm}(s)$. We shall, however, prove in [12] that neither of these factorisations are natural with respect to $V_1, V_2$, which is why we work with the meromorphic $R$–matrices $\mathcal{R}_{V_1,V_2}^{0,\pm}(s)$. 

(iv) $\Gamma_H$ preserves the $q$–characters of Knight and Frenkel–Reshetikhin.
• $\Phi_{\epsilon}$ has an asymptotic expansion of the form $(1+O(s^{-1}))(-s)^{h\Omega}$ in any left half-plane if $\text{Re}(\epsilon h) < 0$, and otherwise in a sector of the form (6.3).

Figure 6.1. Domains of holomorphy and invertibility of $\Phi_{\epsilon}$ (resp. $\Phi_{\epsilon}^-$) given by the ruled region in the right (resp. left) picture, when $\text{Re}(\epsilon h) > 0$. The darker grey region contains poles of $R_0^{0,\epsilon}(s)^{\pm 1}$.

The right and left solution, when defined, are given by the products

$$\Phi_{\epsilon}^+(s) = e^{-h\gamma h\Omega} R_{V_1,V_2}^{0,\epsilon}(s)^{-1} \prod_{m \geq 1} R_{V_1,V_2}^{0,\epsilon}(s + m)^{-1} e^{M\Omega h/m}$$

(6.4)

$$\Phi_{\epsilon}^-(s) = e^{-h\gamma h\Omega} \prod_{m \geq 1} R_{V_1,V_2}^{0,\epsilon}(s - m) e^{M\Omega h/m}$$

(6.5)

6.2. The tensor structure $\mathcal{J}_{V_1,V_2}^{\epsilon}(s)$. Let $\epsilon \in \{\pm\}$ be such that $\epsilon h \notin \mathbb{R}_{<0}$, and $\Phi_{\epsilon}^+(s)$ the right fundamental solution of the abelian $qKZ$ equations (6.1). Define a meromorphic function

$$\mathcal{J}_{V_1,V_2}^{\epsilon} : \mathbb{C} \to GL(V_1 \otimes V_2)$$

by $\mathcal{J}_{V_1,V_2}^{\epsilon}(s) = \Phi_{\epsilon}^+(s + 1)^{-1}$. Thus,

$$\mathcal{J}_{V_1,V_2}^{\epsilon}(s) = e^{h\gamma h\Omega} \prod_{m \geq 1} R_{V_1,V_2}^{0,\epsilon}(s + m) e^{-M\Omega h/m}$$

(6.6)

Theorem.

(i) $\mathcal{J}_{V_1,V_2}^{\epsilon}(s)$ is natural in $V_1, V_2$.

(ii) If $V_1$ and $V_2$ are non-congruent, and $\zeta = e^{2\pi i s}$,

$$\mathcal{J}_{V_1,V_2}^{\epsilon}(s) : \Gamma(V_1) \otimes_{\zeta} \Gamma(V_2) \to \Gamma(V_1 \otimes_{s} V_2)$$

is an isomorphism of $U_q(Lg)$-modules for any $s \notin \sigma(V_2) - \sigma(V_1) + \mathbb{Z}$.
(iii) For any non–congruent $V_1, V_2, V_3 \in \text{Rep}_{\text{id}}(Y_\hbar(g))$, the following is a commutative diagram

\[
\begin{array}{ccc}
\Gamma(V_1) \otimes \zeta_1 \Gamma(V_2) \otimes \zeta_2 \Gamma(V_3) & \xrightarrow{J_{V_1,V_2}(s_1) \otimes 1} & \Gamma(V_1) \otimes \zeta_1 \zeta_2 \Gamma(V_2) \otimes \zeta_2 \Gamma(V_3) \\
\Gamma(V_1 \otimes s_1 V_2) \otimes \zeta_2 \Gamma(V_3) & \xrightarrow{1 \otimes J_{V_2,V_3}(s_2)} & \Gamma(V_1 \otimes \zeta_1 \zeta_2 V_2 \otimes s_2 V_3) \\
\Gamma((V_1 \otimes s_1 V_2) \otimes s_2 V_3) & \xrightarrow{J_{V_1,s_1 V_2,V_3}(s_2)} & \Gamma((V_1 \otimes s_1 V_2) \otimes s_2 V_3)
\end{array}
\]

where $\zeta_i = \exp(2\pi i s_i)$.

(iv) The poles of $J_{V_1,V_2}^+(s)^{\pm 1}$ and $J_{V_1,V_2}^-(s)^{\pm 1}$ are contained in

$$\sigma(V_2) - \sigma(V_1) - Z_{\geq 0} \hbar - \frac{h}{2} \{ l + r \} - Z_{> 0} \quad \text{and} \quad \sigma(V_2) - \sigma(V_1) + Z_{> 0} \hbar - \frac{h}{2} \{ l + r \} - Z_{> 0}$$

where $r$ ranges over the integers such that $c_{ij}^{(r)} \neq 0$ for some $i, j \in \mathbf{I}$.

Remark. Note that the condition $s \not\in \sigma(V_2) - \sigma(V_1) + Z$ is equivalent to $V_1 \otimes s V_2$ being non–congruent, which is required in order to define $\Gamma(V_1 \otimes s V_2)$.

Proof. (i) and (iii)–(iv) follow from (6.6), Theorem 4.9 and Proposition 6.1. (ii) is proved in 6.3.

6.3. Given an element $X \in U_q(Lg)$, we denote its action on $\Gamma(V_1) \otimes \zeta \Gamma(V_2)$ and $\Gamma(V_1 \otimes s V_2)$ by $X'$ and $X''$ respectively. We need to prove that

$$J_{V_1,V_2}^+(s) X' J_{V_1,V_2}^-(s)^{-1} = X''$$

Since $\xi_i(u)$ are group–like with respect to the Drinfeld coproduct, so are the fundamental solutions and the connection matrix of the difference equation $\phi_i(u + 1) = \xi_i(u) \phi_i(u)$, which implies that $\Psi_i(z)' = \Psi_i(z)''$. Since $R^{0,\pm}(s)$ and hence $J^\zeta(s)$ commute with these elements, this proves the required relation for $\{\Psi_i(z)\}_{i \in \mathbf{I}}$. 
We now prove the relation for $\chi^+_{i,k}$. The proof for $\chi^-_{i,k}$ is similar. By 3.2 and 5.3, the action of $(c_i^+)^{-1}\chi^+_{i,k}$ on $\Gamma(V_1) \otimes \zeta \Gamma(V_2)$ is given by

$$
\zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes 1 \, du 
+ \int_{C_2} \Psi_i(\zeta^{-1} w) \otimes \int_{C_2} g_i^+(u) x_i^+(u) \frac{w}{w - e^{2\pi i u} w^{k-1}} \, dwdu 
= \zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes 1 \, du 
+ \int_{C_2} e^{2\pi i k u} g_i^+(u - s) x_i^+(u) \otimes g_i^+(u) x_i^+(u) \, du
$$

where

- $C_1$ encloses $\sigma(V_1)$ and none of its $\mathbb{Z}^*$-translates.
- $C_2$ encloses $C_2, \sigma(\Gamma(V_2)) = \exp(2\pi i \sigma(V_2))$ and none of the points in $\zeta \sigma(\Gamma(V_1)) = \exp(2\pi i (s + \sigma(V_1)))$.

and we used (5.3).

On the other hand, the action of $(c_i^+)^{-1}\chi^+_{i,k}$ on $\Gamma(V_1 \otimes s V_2)$ is given by

$$
\int_{C_{12}} e^{2\pi i k u} g_i^+(u - s) \otimes g_i^+(u) \left( x_i^+(u - s) \otimes 1 + \int_{C_2} \xi_i(v - s) \otimes x_i^+(v) \frac{dv}{u - v} \right) \, du
= \zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes g_i^+(u + s) \, du 
+ \int_{C_2} e^{2\pi i k u} g_i^+(u - s) \xi_i(v - s) \otimes g_i^+(v) x_i^+(v) \, dv
$$

where

- $C_{12}$ encloses $\sigma(V_1 \otimes s V_2) = \sigma(V_1) + s_1 \cup \sigma(V_2)$ and none of its $\mathbb{Z}^*$-translates.
- $C'_2$ encloses $\sigma(V_2)$ and none of the points of $\sigma(V_1) + s$.

$C_1$ is as above, and we assumed that $C_{12}$ encloses $C'_2$, and that $C'_2 = C_2$.

Let us compute the action of $\text{Ad}(\mathcal{J}_{1,V_2}(s))$ on the first summand of $(c_i^+)^{-1}(\chi^+_{i,k})'$. Using Proposition 4.11. and (for $a \in \mathbb{C}$)

$$
\text{Ad}(e^{a i h})(x_i^+(v) \otimes 1) = x_i^+(v) \otimes e^{a i} \xi_i, 0
$$

we get

$$
\text{Ad}(\mathcal{J}_{1,V_2}(s)) \left( \zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes 1 \, du \right)
= \zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes e^{\beta \xi_i, 0} \prod_{n \geq 1} \xi_i(u + s + n) e^{-\beta \xi_i, 0 / n} \, du
= \zeta^k \int_{C_1} e^{2\pi i k u} g_i^+(u) x_i^+(u) \otimes g_i^+(u + s) \, du
$$
by the definition of $g_i^+(u)$ given in (5.2). This yields the first term on the right-hand side of $(c_i^+)^{-1}(\mathcal{X}_{i,k}^+)''$. A similar computation can be carried out for the second summand of $(c_i^+)^{-1}(\mathcal{X}_{i,k}^+)''$ which proves that

$$J^\varepsilon_{V_1,V_2}(s)\mathcal{X}_{i,k}^+(s)^{-1} = (\mathcal{X}_{i,k}^+)''$$

7. The commutative $R$–matrix of the quantum loop algebra

In this section, we review the construction of the commutative part $R^0(\zeta)$ of the $R$–matrix of the quantum loop algebra. We prove that if $|q| \neq 1$, $R^0(\zeta)$ defines a meromorphic commutativity constraint on $\text{Rep}_q(U_q(L\mathfrak{g}))$, when the latter is equipped with the Drinfeld tensor product studied in §3.

7.1. Drinfeld pairing. The Drinfeld pairing for the quantum loop algebra was computed in terms of the loop generators by Damiani [4]. Its restriction to $U^0$ is given in [4, Corollary 9]. Define $\{H_{i,r}\}_{i \in \mathbb{I}, r \in \mathbb{Z}_{\geq 0}}$ by

$$\Psi^\pm_i(z) = \Psi^\pm_{i,0} \exp \left( \pm (q_i - q_i^{-1}) \sum_{r \geq 1} H_{i,\pm r} z^{\mp r} \right)$$

(7.1)

Then, for each $m, n \geq 1$

$$\langle H_{i,m}, H_{j,-n} \rangle = -\delta_{m,n} \frac{q^{mb_{ij}} - q^{-mb_{ij}}}{m(q_i - q_i^{-1})(q_j - q_j^{-1})}$$

(7.2)

where $b_{ij} = d_i a_{ij} = d_j a_{ji}$. Set

$$H^\pm_i(z) = \pm (q_i - q_i^{-1}) \sum_{r \geq 1} H_{i,\pm r} z^{\mp r}$$

Then, by (7.2),

$$\langle H^+_i(z), H^-_j(w) \rangle = \sum_{m \geq 1} \frac{q^{mb_{ij}} - q^{-mb_{ij}}}{m} \left( \frac{w}{z} \right)^m \log \left( \frac{z - q^{-b_{ij}}w}{z - q^{b_{ij}}w} \right)$$

(7.3)

7.2. Construction of $R^0$. We may now follow the argument outlined in §4.2 to construct the canonical element $R^0$ of this pairing. Namely, differentiating (7.3) with respect to $z$ yields

$$\left\langle \frac{d}{dz} H^+_i(z), H^-_j(w) \right\rangle = \frac{1}{z - q^{-b_{ij}}w} - \frac{1}{z - q^{b_{ij}}w} = (T^{b_{ij}} - T^{-b_{ij}}) \frac{1}{z - w}$$

where $Tf(z, w) = f(z, q^{-1}w)$ is the multiplicative shift operator with respect to $w$. Hence, if we define

$$H^{\pm, -}(w) = (T^l - T^{-l})^{-1} \sum_{k \in \mathbb{I}} c_{jk}(T) H^-_k(w) \in wU^0[[w]]$$

(7.4)

where $(T^l - T^{-l})^{-1}$ acts on $w^k$, $k \neq 0$, as multiplication by $(q^{-lk} - q^{lk})^{-1}$, then

$$\left\langle \frac{d}{dz} H^+_i(z), H^{\pm, -}(w) \right\rangle = \delta_{ij} \frac{1}{z - w}$$
Note that $H^j_{-}(w)$ is explicitly given by

$$H^j_{-}(w) = \sum_{k \in I} (q_k - q_k^{-1}) \sum_{n \geq 1} \frac{c_{jk}(q^n)}{q^{ml} - q^{-ml}} H_{k,-n} w^n$$

so that $R^0$ is equal to

$$R^0 = q^{-\Omega_h} \exp \left(- \sum_{i,j \in I, m \geq 1} \frac{m(q_i - q_i^{-1})(q_j - q_j^{-1})c_{ij}(q^m)}{q^{ml} - q^{-ml}} H_{i,m} \otimes H_{j,-m} \right)$$

(7.5)

7.3. $q$–difference equation for $R^0$. Set $R^0(\zeta) = (\tau \otimes 1) R^0$. Then, the following $q$–difference equation is readily seen to hold

$$R^0(q^2 \zeta) R^0(\zeta)^{-1} = \exp \left(- \sum_{i,j \in I, m \geq 1} \frac{m(q_i - q_i^{-1})(q_j - q_j^{-1})c_{ij}(q^m)}{q^{ml} - q^{-ml}} H_{i,m} \otimes H_{j,-m} \right)$$

(7.6)

7.4. Using the method employed in 4.3–4.5, we can show that the right–hand side of (7.6) is the expansion at $\zeta = 0$ of a rational function, once it is evaluated on a tensor product of finite–dimensional representations. It suffices to note that a typical summand can be interpreted as a contour integral as follows

$$\sum_{m \geq 1} \frac{m(q_i - q_i^{-1})(q_j - q_j^{-1})c_{ij}(q^m)}{q^{ml} - q^{-ml}} c^{(r)}_{ij} \int_{\mathbb{C}} \left( \frac{d}{dw} H_i^+(w) \right) \otimes H_j^-(q^{l+r}\zeta w) dw$$

On a tensor product of two finite–dimensional representations $\mathcal{V}_1, \mathcal{V}_2$, the first tensor factor is a rational function of $w$ since

$$\frac{d}{dw} H_i^+(w) = \Psi_i(w)^{-1} \Psi_i'(w)$$

while the second tensor factor can be viewed as a single–valued function defined outside of a cut set. Indeed, $H_j^-(w)$ is a logarithm of the rational $\text{End}(\mathcal{V}_2)$–valued function $\Psi_{j,0}(w)$ which is regular at $w = 0, \infty$ and takes the value 1 at $w = 0$. We will need the following variant of Proposition 4.4.

Proposition. Let $\mathcal{V}$ be a complex, finite–dimensional vector space, and $\psi : \mathbb{C} \to \text{End}(\mathcal{V})$ a rational function, regular at 0 and $\infty$ such that

- $\psi(0) = 1$.
- $[\psi(w), \psi(w')] = 0$ for any $w, w' \in \mathbb{C}$. 

Let $\sigma(\psi) \subset \mathbb{C}^\times$ be the set of poles of $\psi(w)^{\pm 1}$, and define the cut set $Y(\psi)$ by

$$Y(\psi) = \bigcup_{\alpha \in \sigma(\psi)} [\alpha, \infty)$$

where $[\alpha, \infty) = \{ t\alpha : t \in \mathbb{R}_{\geq 1} \}$. Then, there is a unique single-valued, holomorphic function $H(w) = \log_0(\psi(w)) : \mathbb{C} \setminus Y(\psi) \to \text{End}(\mathcal{V})$ such that

$$\exp(H(w)) = \psi(w) \quad \text{and} \quad H(0) = 0$$

Moreover, $[H(w), H(w')] = 0$ for any $w, w' \in \mathbb{C}$ and $H(w)' = \psi(w)^{-1}\psi'(w)$.

The proof of this proposition is analogous to that of Proposition 4.4.

7.5. The operator $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)$. Let $\mathcal{V}_1, \mathcal{V}_2$ be two finite-dimensional representations of $U_q(LG)$. Let $C_1$ be a contour enclosing the set of poles $\sigma(\mathcal{V}_1)$ of $\mathcal{V}_1$, and consider the following operator on $\mathcal{V}_1 \otimes \mathcal{V}_2$

$$\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta) = \exp \left( - \sum_{i,j \in I} \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} \oint_{C_1} \frac{d}{dw} H_i^+(w) \otimes H_j^-(q^{l+r}\zeta w) \, dw \right)$$

where $\zeta \in \mathbb{C}$ is such that $H_j^-(q^{l+r}\zeta w)$ is an analytic function of $w$ within $C_1$ for every $j \in I$ such that $c_{ij}^{(r)} \neq 0$ for some $i \in I$. We have the following counterpart of Theorem 4.5.

Theorem.

(i) $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)$ extends to a rational function of $\zeta$ which is regular at $0$ and $\infty$, and such that $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(0) = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\infty) = 1$.

(ii) The poles of $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)^{\pm 1}$ are contained in $\sigma(\mathcal{V}_2)\sigma(\mathcal{V}_1)^{-1}q^{-l-r}$, where $r$ ranges over the integers such that $c_{ij}^{(r)} \neq 0$ for some $i, j \in I$.

(iii) For any $\zeta, \zeta'$ we have $[\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta), \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta')] = 0$.

(iv) For any $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \text{Rep}_{\text{id}}(U_q(LG))$, we have

$$\mathcal{A}_{\mathcal{V}_1 \otimes \mathcal{V}_2, \mathcal{V}_3}(\zeta_2) = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_3}(\zeta_1 \zeta_2) \mathcal{A}_{\mathcal{V}_2, \mathcal{V}_3}(\zeta_2)$$

$$\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2 \otimes \mathcal{V}_3}(\zeta_1 \zeta_2) = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_3}(\zeta_1 \zeta_2) \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta_1)$$

(v) The following shifted unitary condition holds

$$\sigma \circ \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta^{-1}) \circ \sigma^{-1} = \mathcal{A}_{\mathcal{V}_2, \mathcal{V}_1}(q^{-2l} \zeta)$$

where $\sigma : \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_2 \otimes \mathcal{V}_1$ is the flip of the tensor factors.

(vi) For every $\alpha, \beta \in \mathbb{C}^\times$ we have

$$\mathcal{A}_{\mathcal{V}_1(\alpha), \mathcal{V}_2(\beta)}(\zeta) = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta \alpha \beta^{-1})$$
7.6. Commutation relation with $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$. Let $\mathcal{C} \subset \mathbb{C}$ be a contour, and $a_\ell : \mathbb{C} \to \text{End}(\mathcal{V}_\ell)$, $\ell = 1, 2$ two meromorphic functions which are analytic within $\mathcal{C}$ and commute with the operators $\{\Psi^{\pm}_{i,\pm r}\}_{i \in \mathbb{I}, r \in \mathbb{N}}$. For any $k \in \mathbb{I}$, define operators $X_k^{\pm,\ell} \in \text{End}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ by

$$X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(w)X_k^{\pm}(w) \otimes a_2(w) \, dw \quad \text{and} \quad X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(w) \otimes a_2(w)X_k^{\pm}(w) \, dw$$

**Proposition.** The following commutation relations hold

$$\text{Ad}(\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta))X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(w)X_k^{\pm}(w) \otimes a_2(w) \Psi_k(q^{2l}\zeta w)^{\pm 1} \Psi_k(\zeta w) \, dw$$

$$\text{Ad}(\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta))X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(w)X_k(\zeta^{-1}w)^{\pm 1} \Psi_k(q^{-2l}\zeta^{-1}w) \otimes a_2(w)X_k^{\pm}(w) \, dw$$

The proof of this proposition is identical to that of Proposition 4.10, except that the following version of relation (2.1) is needed. For each $i, k \in \mathbb{I}$,

$$[\Psi_i(z)^{-1}\Psi_i'(z), X_k^{\pm}(w)] = \left(\frac{1}{z - q^{\pm b_k}w} - \frac{1}{z - q^{\pm b_k}w}\right)X_k^{\pm}(w)$$

$$+ \frac{wq^{\pm b_k}}{z(z - q^{\pm b_k}w)}X_k^{\pm}(q^{\mp b_k}z) - \frac{w}{z(q^{\pm b_k}z - w)}X_k^{\pm}(q^{\pm b_k}z)$$

One can derive this relation easily from (QL3) of Proposition 2.8 following the computation given in the proof of Lemma 2.13.

7.7. The abelian $R$–matrix of $U_q(L\mathfrak{g})$. Assume now that $|q| \neq 1$. Let $\mathcal{V}_1, \mathcal{V}_2 \in \text{Rep}_{fd}(U_q(L\mathfrak{g}))$, and let $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ be the operator defined in 7.5. Consider the $q$–difference equation

$$\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(q^{2l}\zeta) = \mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$$

This equation is regular at 0 and $\infty$, in that $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(0) = 1 = \mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(\infty)$. By the general theory of $q$–difference equations (see e.g., [22]), it admits two unique formal solutions $\mathcal{A}_\pm(\zeta)$ near $q^{\pm \infty}$, which are normalized by

$$\mathcal{A}_+^+(q^{\infty}) = 1 = \mathcal{A}_-^-(q^{-\infty})$$

These solutions converge in a neighborhood of $q^{\pm \infty}$, and extend to meromorphic functions on the entire complex plane which are given by the products

$$\mathcal{A}_+ = \prod_{n \geq 0} \mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(q^{2in}\zeta) \quad \text{and} \quad \mathcal{A}_- = \prod_{n \geq 1} \mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}(q^{-2in}\zeta)$$

Set

$$\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}^{0,\pm}(\zeta) = q^{\varepsilon \Omega} \mathcal{A}_\pm(\zeta)$$

where $\varepsilon = +$ if $q^{\pm \infty} = 0$ and $\varepsilon = -$ if $q^{\pm \infty} = \infty$. By uniqueness, the evaluation on $\mathcal{V}_1 \otimes \mathcal{V}_2$ of the operator $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}^{0,\varepsilon}(\zeta)$ given by (7.5) is the expansion near $\zeta = 0$ of $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}^{0,\varepsilon}(\zeta)$, where $\varepsilon \in \{\pm\}$ is such that $q^{\varepsilon \infty} = 0$.

The following is the analog of Theorem 4.9 for $U_q(L\mathfrak{g})$.

**Theorem.** The operators $\mathcal{A}_{\mathcal{V}_1,\mathcal{V}_2}^{0,\pm}(\zeta)$ have the following properties
(i) The map
\[ \sigma \circ R_{V_1, V_2}^{0,\pm}(\zeta) : V_1(\zeta) \otimes V_2 \rightarrow V_2 \otimes V_1(\zeta) \]
where \( \sigma \) is the flip of tensor factors, is a morphism of \( U_q(\mathfrak{g}) \)-modules, which is natural in \( V_1 \) and \( V_2 \).

(ii) For any \( V_1, V_2, V_3 \in \text{Rep}_{\mathfrak{sl}}(U_q(\mathfrak{g})) \) we have
\[
R_{V_1 \otimes V_2, V_3}(\zeta_2) = R_{V_1, V_3}(\zeta_1 \zeta_2) R_{V_2, V_3}(\zeta_2)
\]
\[
R_{V_1, V_2 \otimes V_3}(\zeta_1 \zeta_2) = R_{V_1, V_3}(\zeta_1 \zeta_2) R_{V_1, V_2}(\zeta_1)
\]

(iii) The following unitary condition holds
\[ \sigma \circ R_{V_1, V_2}^{0,\pm}(\zeta^{-1}) \circ \sigma^{-1} = R_{V_2, V_1}^{0,\pm}(\zeta) \]

(iv) For \( \alpha, \beta \in \mathbb{C}^\times \), we have
\[ R_{V_1(\alpha), V_2(\beta)}^{0,\pm}(\zeta) = R_{V_1, V_2}^{0,\pm}(\zeta \alpha \beta^{-1}) \]

(v) For any \( \zeta, \zeta' \),
\[ [R_{V_1, V_2}^{0,\pm}(\zeta), R_{V_1, V_2}^{0,\pm}(\zeta')] = 0 = [R_{V_1, V_2}^{0,\pm}(\zeta), R_{V_1, V_2}^{0,\mp}(\zeta')] \]

(vi) \( R_{V_1, V_2}^{0,\pm}(\zeta) \) is holomorphic near \( q^{\pm \infty} \), and
\[ R_{V_1, V_2}^{0,\pm}(q^{\pm \infty}) = \begin{cases} q^{\Omega_h} & \text{if } q^{\pm \infty} = 0 \\
q^{-\Omega_h} & \text{if } q^{\pm \infty} = \infty \end{cases} \]

(vii) The poles of \( R_{V_1, V_2}^{0,+,\pm}(\zeta)^{\pm 1} \) and \( R_{V_1, V_2}^{0,-,\pm}(\zeta)^{\pm 1} \) are contained in
\[ \sigma(V_2) \sigma(V_1)^{-1} q^{-l-r} q^{-2lZ_{>0}} \quad \text{and} \quad \sigma(V_2) \sigma(V_1)^{-1} q^{-l-r} q^{2lZ_{>0}} \]
respectively, where \( r \) ranges over the integers such that \( c_{ij}^{(r)} \neq 0 \) for some \( i, j \in I \).

8. Kohno–Drinfeld theorem for abelian, additive \( q \)KZ equations

In this section, we prove that, when \( \text{Im} \ h \neq 0 \), the monodromy of the additive \( q \)KZ equations on \( n \) points defined by the commutative \( R \)-matrix of the Yangian is given by the commutative \( R \)-matrix of the quantum loop algebra. The general case is treated in 8.5, and follows from the \( n = 2 \) case which is studied in 8.2. In turn, the latter rests on relating the coefficient matrices of the difference equations whose monodromy are the commutative \( R \)-matrix of \( Y_h(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \) respectively, which is done in 8.1 below.
8.1. Let $V_1, V_2$ be two finite-dimensional representations of $Y_h(g)$, $A_{V_1, V_2}(s)$ the meromorphic $GL(V_1 \otimes V_2)$-valued function constructed in 4.5, and consider the difference equation

$$f(s + 1) = A_{V_1, V_2}(s)f(s) \quad (8.1)$$

Assume further that $V_1, V_2$ are non-congruent, let $\mathcal{V}_\ell = \Gamma(V_\ell)$ be the representations of $U_q(Lg)$ obtained by using the functor $\Gamma$ of Section 5, and $\mathcal{A}_{V_1, V_2}(\zeta) \in GL(V_1 \otimes V_2)$ the operator constructed in 7.5.

**Proposition.** The operator $\mathcal{A}_{V_1, V_2}(\zeta)$ is the monodromy of the difference equation (8.1). That is,

$$\mathcal{A}_{V_1, V_2}(\zeta) = \prod_{m \in \mathbb{Z}} A_{V_1, V_2}(s + m) \bigg|_{\zeta = e^{2\pi i s}}$$

**Proof.** For the purposes of the proof, we restrict ourselves to a typical factor in the definition of $A_{V_1, V_2}(s)$. That is, fix $i, j \in I$ and define

$$A_{V_1, V_2}^{ij}(s) = \exp \left( \oint_{\mathcal{C}_1} t'_i(v) \otimes t_j(v + s) \, dv \right)$$

where $\mathcal{C}_1$ encloses the poles of $\xi_i(v)^{\pm 1}$ on $V_1$, and $s$ is such that $t_j(v + s)$ is analytic within $\mathcal{C}_1$. Since $V_1$ is non-resonant, we may further assume that $\mathcal{C}_1$ contains none of the $\mathbb{Z}^*$-translates of the poles of $\xi_i(v)^{\pm 1}$ on $V_1$.

By Theorem 4.5, $A_{V_1, V_2}^{ij}(s)$ is a rational function of the form $1 + O(s^{-2})$.

The corresponding monodromy matrix $M(\zeta)$ is a rational function of $\zeta = e^{2\pi i s}$ which is given by

$$M(\zeta) = \prod_{m \in \mathbb{Z}} A_{V_1, V_2}^{ij}(s + m) \bigg|_{\zeta = e^{2\pi i s}}$$

The corresponding factor of $\mathcal{A}_{V_1, V_2}(\zeta)$ is given by

$$\mathcal{A}_{V_1, V_2}^{ij}(\zeta) = \exp \left( \oint_{\mathcal{C}_1} \Psi_i(w)^{-1} \Psi'_j(w) \otimes H^{-}_j(w\zeta) \, dw \right)$$

where $\mathcal{C}_1 = \exp(2\pi i \mathcal{C}_1)$, and $H^{-}_j(w) = \log_0(\Psi_j, 0\Psi_j(w))$ is given by Proposition 7.4.

We wish to show that $M(\zeta) = \mathcal{A}_{V_1, V_2}^{ij}(\zeta)$. Since both sides are rational functions of $\zeta$, it suffices to prove this for $\zeta$ near 0, that is $\text{Im}(s) \gg 0$. Now

$$M(\zeta) = \lim_{N \to \infty} \exp \left( \sum_{m = -N}^{N} \oint_{\mathcal{C}_1} t'_i(v) \otimes t_j(v + s + m) \, dv \right)$$

---

9Note that (8.1) differs from the difference equation considered in 4.8 since its step is 1, not $l h$. 
Since \( t_j(v) = h \xi_j v^{-1} + O(v^{-2}) \), the sum \( \sum_{m=-N}^N t_j(u + m) \) converges uniformly on compact subsets of \( \{|\Im u| > R\} \) for \( R \) large enough. Its exponential is \( \prod_m \xi_j(u + m) = \Psi_j(e^{2\pi i u}) \) (see §5.3). By the uniqueness of \( \log_0 \) this implies that, for \( \Im s \gg 0 \),

\[
\lim_{N \to \infty} \sum_{m=-N}^N t_j(v + s + m) = H_j^- (\xi e^{2\pi i u}) + \pi i h \xi_j,0
\]

so that

\[
M(\zeta) = \exp \left( \oint_{C_1} t'_i(v) \otimes (H_j^- (\xi e^{2\pi i u}) + \pi i h \xi_j,0) \, dv \right)
\]

Since \( t'_i(v) = O(v^{-2}) \), we get \( \oint t'_i(v) \otimes \xi_j,0 \, dv = 0 \), which implies that

\[
M(\zeta) = \exp \left( \oint_{C_1} \xi_i(v)^{-1} \xi_i(v)' \otimes H_j^- (\xi e^{2\pi i u}) \, dv \right)
\]

Noting that, by (5.3)

\[
\Psi_i(e^{2\pi i u})^{-1} \frac{d\Psi_i(e^{2\pi i u})}{du} = g_i^+(v)^{-1} g_i^+(v)' + \xi_i(v)^{-1} \xi_i(v)' + g_i^-(v)^{-1} g_i^-(v)'
\]

and that \( g_i^\pm(v) \) are analytic and invertible within \( C_1 \) by the non–congruence assumption, so that

\[
\oint_{C_1} g_i^\pm(v)^{-1} g_i^\pm(v)' \otimes H_j^- (\xi e^{2\pi i u}) \, dv = 0
\]

we get

\[
M(\zeta) = \exp \left( \oint_{C_1} \Psi_i(v)^{-1} \frac{d\Psi_i(v)}{dv} \otimes H_j^- (\xi e^{2\pi i u}) \, dv \right)
\]

\[
= \exp \left( \oint_{C_1} \Psi_i(v)^{-1} \frac{d\Psi_i(v)}{dv} \otimes H_j^- (\xi v) \, dv \right)
\]

as claimed. \( \square \)

8.2. The (reduced) qKZ equations on \( n = 2 \) points. Assume henceforth that \( \Im h \neq 0 \). Fix \( \varepsilon \in \{\pm\} \), let \( V_1, V_2 \in \Rep_{td}(Y_h(g)) \), and consider the abelian qKZ equation

\[
f(s + 1) = R_{V_1, V_2}^{0, \varepsilon}(s) f(s)
\]

with values in \( V_1 \otimes V_2 \).

By Proposition 6.1, this equation admits both a right and a left canonical solution \( \Phi_{\pm}(s) \). The corresponding connection matrix is given by

\[
S_{V_1, V_2}^{\varepsilon}(s) = \Phi_{+}^{-1}(s) \Phi_{-}(s)
\]

\[
= \lim_{N \to \infty} R_{V_1, V_2}^{0, \varepsilon}(s + N) \cdots R_{V_1, V_2}^{0, \varepsilon}(s + N) \cdots R_{V_1, V_2}^{0, \varepsilon}(s - N)
\]
and is a meromorphic function of $\zeta = e^{2\pi i s}$ which admits a limit as $\text{Im } s \to \pm \infty$, depending on whether $\text{Im}(\varepsilon h) \geq 0$. Thus, $\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta)$ is regular at $\zeta = q^{\varepsilon \infty}$, and such that

$$\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta) = \begin{cases} q^\Omega_h & \text{if } q^{\varepsilon \infty} = 0 \\ q^{-\Omega_h} & \text{if } q^{\varepsilon \infty} = \infty \end{cases}$$

The following equates the monodromy of the abelian $q$KZ equations with the commutative $R$–matrix of $U_q(L\mathfrak{g})$ constructed in 7.7.

**Theorem.** If $V_1, V_2$ are non–congruent, $V_\varepsilon = \Gamma(V_\varepsilon)$ are the corresponding representations of $U_q(L\mathfrak{g})$, and $\mathcal{R}^{0, \varepsilon}_{V_1, V_2}(\zeta)$ is the commutative $R$–matrix of $U_q(L\mathfrak{g})$, then

$$\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta) = \mathcal{R}^{0, \varepsilon}_{V_1, V_2}(\zeta)$$

**Proof.** Let $\mathcal{A}_\pm(s)$ be the right and left fundamental solutions of the difference equation $f(s+1) = \mathcal{A}_{V_1, V_2}(s)f(s)$ considered in 8.1. We claim that

$$\Phi^\varepsilon_+(s + \varepsilon h) \Phi^\varepsilon_-(s) = \mathcal{A}_\pm(s)$$  \hspace{1cm} (8.2)

Assuming this for now, we see that $\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta)$ and $\mathcal{R}^{0, \varepsilon}_{V_1, V_2}(\zeta)$ satisfy the same $q$–difference equation. Indeed,

$$\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta) = \mathcal{R}^{0, \varepsilon}_{V_1, V_2}(\zeta)$$

where the third equality follows by Proposition 8.1, and the last one by definition of $\mathcal{R}^{0, \varepsilon}_{V_1, V_2}$. Since both $\mathcal{S}_{V_1, V_2}^\varepsilon$ and $\mathcal{R}^{0, \varepsilon}_{V_1, V_2}$ are holomorphic near, and have the same value at $\zeta = q^{\varepsilon \infty}$, they are equal.

Returning to the claim, let $L_\pm^\varepsilon(s)$ denote the left–hand side of (8.2). Then,

$$L_\pm^\varepsilon(s + 1) = \Phi^\varepsilon_+(s + \varepsilon h + 1) \Phi^\varepsilon_-(s + 1) \Phi^\varepsilon_+(s + \varepsilon h) \Phi^\varepsilon_-(s)$$

$$= \mathcal{R}^{0, \varepsilon}_{V_1, V_2}(s + \varepsilon h) \mathcal{R}^{0, \varepsilon}_{V_1, V_2}(s)$$

Thus, $L_\pm^\varepsilon(s)$ and $\mathcal{A}_\pm(s)$ satisfy the same difference equation. Since they also have the same asymptotics as $s \to \infty$ by Proposition 6.1, it follows that they are equal. \hfill \Box

**Remark.** The monodromy $\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta)$ may be written in terms of the tensor structures $\mathcal{J}^\varepsilon_{V_1, V_2}(s)$ constructed in 6.2 as

$$\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta) = \mathcal{J}^\varepsilon_{V_1, V_2}(s) \mathcal{R}^{0, \varepsilon}_{V_1, V_2}(s) \left( \mathcal{J}^{\varepsilon}_{V_2, V_1}(-s) \right)^{-1}_{21}$$

where we used the unitarity constraint (iii) of Theorem 4.9. This, and Theorem 8.2 show that these tensor structures intertwine the meromorphic braidings on $\text{Rep}_{\text{id}}(V_h(\mathfrak{g}))$ and $\text{Rep}_{\text{id}}(U_q(L\mathfrak{g}))$ given by $\mathcal{R}^{0, \varepsilon}_{V_1, V_2}(s)$ and $\mathcal{S}_{V_1, V_2}^\varepsilon(\zeta)$. 


8.3. The abelian qKZ equations. Fix $\varepsilon \in \{\pm 1\}$ and $n \geq 2$, and let $V_1, \ldots, V_n \in \text{Rep}_{\text{eq}}(Y_h(g))$.

The following system of difference equations for a meromorphic function of $n$ variables $\Phi : \mathbb{C}^n \to \text{End}(V_1 \otimes \cdots \otimes V_n)$ is an abelian version of the qKZ equations [10, 23]

$$\Phi(s + e_i) = A_i(s)\Phi(s)$$  \hspace{1cm} (8.3)

where $s = (s_1, \ldots, s_n)$, $\{e_i\}_{i=1}^n$ are the standard basis of $\mathbb{C}^n$, and

$$A_i(s) = \mathcal{R}_{i-1,i}^{0,\varepsilon}(s_{i-1} - s_i - 1)^{-1} \cdots \mathcal{R}_{1,i}^{0,\varepsilon}(s_1 - s_i - 1)^{-1} \cdot \mathcal{R}_{i,n}^{0,\varepsilon}(s_i - s_n) \cdots \mathcal{R}_{i,i+1}^{0,\varepsilon}(s_i - s_{i+1})$$

with $\mathcal{R}_{i,j}^{0,\varepsilon} = \mathcal{R}_{V_i^j}$.

Since $\mathcal{R}_{i,j}^{0,\varepsilon}$ satisfies the Yang–Baxter equations on $V_i \otimes V_j \otimes V_k$

$$\mathcal{R}_{i,j}^{0,\varepsilon}(u)\mathcal{R}_{i,k}^{0,\varepsilon}(u + v)\mathcal{R}_{j,k}^{0,\varepsilon}(v) = \mathcal{R}_{i,k}^{0,\varepsilon}(v)\mathcal{R}_{i,j}^{0,\varepsilon}(u + v)\mathcal{R}_{j,k}^{0,\varepsilon}(u)$$

the above system is integrable, that is satisfies

$$A_i(s + e_j)A_j(s) = A_j(s + e_i)A_i(s)$$

8.4. Canonical fundamental solutions. The above system admits a set of canonical fundamental solutions which are parametrised by permutations $\sigma \in \mathfrak{S}_n$, and correspond to the right/left solutions in the case $n = 2$.

To describe them, let $\Sigma^{\varepsilon}_{\pm,ij} \subset \mathbb{C}^n$ denote the asymptotic zones given in Proposition 6.1 with $s = s_i - s_j$, where $1 \leq i \neq j \leq n$. Thus,

$$\Sigma^{\varepsilon}_{\pm,ij} = \{s \in \mathbb{C}^n | \pm \text{Re}(s_i - s_j) \gg 0 \text{ and } \pm \text{Re}((s_i - s_j)/n) \gg 0\}$$

where $n \in \mathbb{C}^\times$ is perpendicular to $h$ and such that $\text{Re}(n) \geq 0$, and the second condition in the definition of $\Sigma^{\varepsilon}_{\pm,ij}$ is required only if $\pm \text{Re}(\varepsilon h) < 0$.

For a permutation $\sigma \in \mathfrak{S}_n$, set

$$C^{\pm}(\sigma) = \{i < j | \sigma^{-1}(i) \leq \sigma^{-1}(j)\}$$

and define $\Sigma^{\varepsilon}(\sigma) \subset \mathbb{C}^n$ by

$$\Sigma^{\varepsilon}(\sigma) = \bigcap_{(i,j) \in C^+(\sigma)} \Sigma^{\varepsilon}_{+,ij} \cap \bigcap_{(i,j) \in C^-(\sigma)} \Sigma^{\varepsilon}_{-,ij}$$

**Proposition.** For any $\sigma \in \mathfrak{S}_n$, the equation (8.3) admits a fundamental solution $\Phi^{\varepsilon}_{\sigma}$ which is uniquely determined by the following requirements

1. $\Phi^{\varepsilon}_{\sigma}$ is holomorphic and invertible in $Z^\varepsilon(\sigma)$.
2. $\Phi^{\varepsilon}_{\sigma}$ has an asymptotic expansion of the form

$$\Phi^{\varepsilon}_{\sigma}(s) \sim (1 + O(1)) \prod_{(i,j) \in C^+(\sigma)} (s_i - s_j)^{M_{ij}} \prod_{(i,j) \in C^-(\sigma)} (s_j - s_i)^{M_{ij}}$$

for $s \in \Sigma^{\varepsilon}(\sigma)$, with $s_i - s_j \to \infty$ for any $i \neq j$. 

Proof. The solution $\Phi_\sigma^\varepsilon$ is constructed as follows. For each $i < j$, let $\Phi_{\pm,ij}^\varepsilon$ be the right and left canonical solutions of the abelian $q$KZ equation $\Phi_{ij}(s + 1) = R_{ij}(s)\Phi_{ij}(s)$ given in Proposition 6.1. Then,

$$\Phi_\sigma^\varepsilon = \prod_{(i,j) \in C^+(\sigma)} \Phi_+^\varepsilon(s_i-s_j) \prod_{(i,j) \in C^-(\sigma)} \Phi_-^\varepsilon(s_i-s_j)$$

We now sketch the uniqueness (see, e.g., [11, §4.3] for the one variable case). The difference $\Xi_\varepsilon$ is holomorphic for $s \in \Sigma^\varepsilon(\sigma)$, and periodic under the lattice $\mathbb{Z}^n \subset \mathbb{C}^n$. It therefore descends to a holomorphic function on the torus $T = \mathbb{C}^n/\mathbb{Z}^n$. The asymptotics of $\Phi_\sigma^\varepsilon$ and $\Psi_\sigma^\varepsilon$ show that $\Xi_\varepsilon$ extends to a holomorphic function on the toric compactification of $T$ corresponding to the fan generated by the $A_{n-1}$ arrangement, which takes the value 1 at each of the points at infinity corresponding to the chambers of that arrangement. It follows that $\Xi_\varepsilon(\sigma) \equiv 1$.

8.5. Kohno–Drinfeld theorem for abelian $q$KZ equations. Assume now that $V_1, \ldots, V_n$ are non–resonant, and let $\mathcal{V}_i = \Gamma(V_i)$ be the corresponding representations of $U_q(L\mathfrak{g})$. The following computes the monodromy of the abelian $q$KZ equations on $V_1 \otimes \cdots \otimes V_n$ in terms of the commutative $R$–matrix of $U_q(L\mathfrak{g})$ acting on $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$.

Theorem. Let $\sigma \in \mathfrak{S}_n$, and set $\sigma_i = (i \ i + 1)$. Then,

$$(\Phi_\sigma^\varepsilon)^{-1} \Phi_\sigma^\delta = \mathcal{R}_{\mathcal{V}_i,\mathcal{V}_{i+1}}^0(\zeta_{i+1}^{-1})^{-1}$$

if $(i,i+1) \in C^+(\sigma)$, where $\zeta_j = e^{2\pi i s_j}$.

Proof. This follows from the explicit form of the canonical fundamental solutions given by Proposition 8.4 and Theorem 8.2.

Appendix A. The inverse of the $T$–Cartan matrix of $\mathfrak{g}$

A.1. Let $A = (a_{ij})_{i \in \mathbf{I}}$ be a Cartan matrix of finite type, and $d_i \in \mathbb{N}^\times$ ($i \in \mathbf{I}$) be relatively prime symmetrising integers, i.e., $d_i a_{ij} = d_j a_{ji}$ for every $i, j \in \mathbf{I}$. Consider the symmetrised Cartan matrix $B = (d_i a_{ij})$, and its $q$–analog $B(q) = ([d_i a_{ij}]_q)$. The latter defines a $\mathbb{C}(q)$–valued, symmetric bilinear form by

$$(\alpha_i, \alpha_j) = [d_i a_{ij}]_q$$

We give below an explicit expressions for the fundamental coweights $\{\lambda_i^\vee(q)\}_{i \in \mathbf{I}}$ in terms of $\{\alpha_i\}$. That is, we compute certain elements $\lambda_i^\vee(q) \in \sum_{j \in \mathbf{I}} \mathbb{Q}(q)\alpha_j$ such that $\lambda_i^\vee(q), \alpha_j = \delta_{ij}$ for every $i, j \in \mathbf{I}$. The main result of these calculation is the following.

Theorem. Let $l = mh^\vee$ where $m = 1, 2, 3$ for types ADE, BCF and G, respectively, and $h^\vee$ is the dual Coxeter number. Then, for each $i \in \mathbf{I}$

$$[l]_q \lambda_i^\vee(q) \in \oplus_{j \in \mathbf{I}} \mathbb{N}[q,q^{-1}]\alpha_j$$
\[ [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} = \sum_{i=0}^{m-1} q^{m-1-2i} \in \mathbb{N}[q, q^{-1}] \]. Moreover, define \( \{m\}_q := q^m + q^{-m} \). The following identity is immediate and will be needed later.

\[ [a]_q [b]_q = [a + b]_q + [a - b]_q \] (A.1)

which belongs to \( \mathbb{N}[q, q^{-1}] \) if \( a \geq b \).

Also we note that for \( a, b \in \mathbb{N} \) we have

\[ \frac{[ab]_q}{[a]_q} = [b]_q \in \mathbb{N}[q, q^{-1}] \]

A.3. \( A_n \). In this case \( l = n + 1 \). We have

\[ \lambda_i^\gamma(q) = \frac{1}{[n+1]_q} \left( [n - i + 1]_q \left( \sum_{j=1}^{i-1} [j]_q \alpha_j \right) + [i]_q \left( \sum_{j=i}^n [n - j + 1]_q \alpha_j \right) \right) \]

Thus the assertion of Theorem A.1 holds in this case.

A.4. \( B_n \). In this case \( l = 2(n + 1) \). For \( 1 \leq i \leq n - 1 \) we have

\[ \lambda_i^\gamma(q) = \frac{1}{\{n+1\}_q} \left( [n - i + 1]_q \left( \sum_{j=1}^{i-1} [j]_q \alpha_j \right) + [i]_q \left( \sum_{j=i}^{n-1} [n - j + 1]_q \alpha_j \right) + \alpha_n \right) \]

and

\[ \lambda_n^\gamma(q) = \frac{1}{\{n+1\}_q} \left( \sum_{j=1}^{n-1} [j]_q \alpha_j \right) + \frac{[n]_q}{[2]_q} \alpha_n \]

The statement of Theorem A.1 in this case follows for \( 1 \leq i \leq n - 1 \) from the identity \( [m]_q \{m\}_q = [2m]_q \). For \( \lambda_n^\gamma(q) \), we can write (using the same identity)

\[ \lambda_n^\gamma(q) = \frac{1}{[2(n+1)]_q} \left( [n + 1]_q \left( \sum_{j=1}^{n-1} [j]_q \alpha_j \right) + \frac{[n]_q [n]_q}{[2]_q} \alpha_n \right) \]

Now it is clear that the coefficient of \( \alpha_n \) is a Laurent polynomial in \( q \) with positive integer coefficients.

A.5. \( C_n \). In this case \( l = 2(2n - 1) \). We have the following for each \( 1 \leq i \leq n - 1 \)

\[ \lambda_i^\gamma(q) = \frac{1}{[2]_q \{2n - 1\}_q} \left( [2n - 2i - 1]_q \left( \sum_{j=1}^{i-1} [j]_q \alpha_j \right) + [i]_q \left( \sum_{j=i}^{n-1} [2n - 2j - 1]_q \alpha_j \right) + [2]_q \alpha_n \right) \]
and
\[
\lambda_n^\vee(q) = \frac{1}{[2]_q [2n-1]_q} \sum_{j=1}^{n} [2j]_q \alpha_j
\]

The statement of Theorem A.1 follows for \( \lambda_n^\vee(q) \). For \( 1 \leq i \leq n - 1 \) we will have to use the following variant of (A.1):
\[
\frac{[2n-1]_q [2n-2j-1]_q}{[2]_q} = \frac{[4n-2j-2]_q + [2j]_q}{[2]_q} \in \mathbb{N}[q, q^{-1}]
\]

A.6. \( D_n \). In this case \( l = 2n - 2 \). We have the following for \( 1 \leq i \leq n - 2 \):
\[
\lambda_i^\vee(q) = \frac{1}{\{n-1\}_q} \left( \{n-i-1\}_q \left( \sum_{j=1}^{i-1} [j]_q \alpha_j \right) + [i]_q \left( \sum_{j=i}^{n-2} \{n-j-1\}_q \alpha_j \right) + \alpha_{n-1} + \alpha_n \right)
\]

and
\[
\lambda_{n-1}^\vee(q) = \frac{1}{\{n-1\}_q} \left( \sum_{j=1}^{n-2} [j]_q \alpha_j \right) + \frac{[n]_q}{[2]_q} \alpha_{n-1} + \frac{[n-2]_q}{[2]_q} \alpha_n
\]
\[
\lambda_{n-1}^\vee(q) = \frac{1}{\{n-1\}_q} \left( \sum_{j=1}^{n-2} [j]_q \alpha_j \right) + \frac{[n]_q}{[2]_q} \alpha_{n-1} + \frac{[n-2]_q}{[2]_q} \alpha_n
\]

Again we obtain Theorem A.1 by the same argument as for \( B_n \).

A.7. \( F_4 \). In this case \( l = 18 \). We get the following
\[
\lambda_1^\vee(q) = \frac{\{3\}_q}{\{9\}_q} \left( \{5\}_q \alpha_1 + [3]_q \alpha_2 + [2]_q \alpha_3 + \alpha_4 \right)
\]
\[
\lambda_2^\vee(q) = \frac{\{3\}_q}{\{9\}_q} \left( [3]_q \alpha_1 + [6]_q \alpha_2 + [4]_q \alpha_3 + [2]_q \alpha_4 \right)
\]
\[
\lambda_3^\vee(q) = \frac{\{3\}_q}{\{9\}_q} \left( \{2\}_q \{3\}_q \alpha_1 + [4]_q \{3\}_q \alpha_2 + [3]_q^2 \{2\}_q \alpha_3 + \alpha_4 \right)
\]
\[
\lambda_4^\vee(q) = \frac{\{3\}_q}{\{9\}_q} \left( \{3\}_q \alpha_1 + [2]_q \{3\}_q \alpha_2 + [3]_q^2 \alpha_3 + \frac{\{3\}_q \{4\}_q}{[2]_q} \alpha_4 \right)
\]

Again the statement of Theorem A.1 is clearly true, except for the coefficient of \( \alpha_4 \) in \( \lambda_4^\vee(q) \). For that entry we have
\[
\frac{[9]_q \{3\}_q}{[2]_q} = \frac{[12]_q + [6]_q}{[2]_q} \in \mathbb{N}[q, q^{-1}]
\]

A.8. \( G_2 \). In this case \( l = 12 \). We have the following answer
\[
\lambda_1^\vee(q) = \frac{\{2\}_q}{\{6\}_q} \left( \frac{[2]_q}{[3]_q} \alpha_1 + \alpha_2 \right) \quad \lambda_2^\vee(q) = \frac{\{2\}_q}{\{6\}_q} \left( \alpha_1 + \{3\}_q \alpha_2 \right)
\]

As before we multiply and divide these expressions by \([6]_q\) to get the denominator \([12]_q\). Then it is easy to see the coefficients of \( \alpha_1, \alpha_2 \) are in \( \mathbb{N}[q, q^{-1}] \) as claimed.
A.9. E series. The computations below were carried out using sage.

A.10. E6. In this case \( l = 12 \). We have the following expressions:

\[
\begin{align*}
[12]_q \lambda^1_0(q) &= \{3\}_q [8]_q \alpha_1 + \{2\}_q [6]_q \alpha_2 + \{2\}_q \{3\}_q [5]_q \alpha_3 + [4]_q [6]_q \alpha_4 \\
&\quad + [2]_q \{3\}_q [4]_q \alpha_5 + \{3\}_q [4]_q \alpha_6 \\
[12]_q \lambda^2_0(q) &= \{2\}_q [6]_q \alpha_1 + \{2\}_q \{3\}_q [6]_q \alpha_2 + [4]_q [6]_q \alpha_3 + \{2\}_q [3]_q [6]_q \alpha_4 \\
&\quad + [4]_q [6]_q \alpha_5 + \{2\}_q [6]_q \alpha_6 \\
[12]_q \lambda^3_0(q) &= \{2\}_q \{3\}_q [5]_q \alpha_1 + [4]_q [6]_q \alpha_2 + \{3\}_q [4]_q [5]_q \alpha_3 + \{1\}_q [4]_q [6]_q \alpha_4 \\
&\quad + [2]_q^2 \{3\}_q [4]_q \alpha_5 + [2]_q \{3\}_q [4]_q \alpha_6 \\
[12]_q \lambda^4_0(q) &= [4]_q [6]_q \alpha_1 + \{2\}_q \{3\}_q [6]_q \alpha_2 + \{2\}_q [4]_q [6]_q \alpha_3 + [3]_q [4]_q [6]_q \alpha_4 \\
&\quad + [2]_q [4]_q [6]_q \alpha_5 + [4]_q [6]_q \alpha_6 \\
[12]_q \lambda^5_0(q) &= [2]_q \{3\}_q [4]_q \alpha_1 + [4]_q [6]_q \alpha_2 + [2]_q^2 \{3\}_q [4]_q \alpha_3 + [2]_q [4]_q [6]_q \alpha_4 \\
&\quad + \{3\}_q [4]_q [5]_q \alpha_5 + [2]_q \{3\}_q [5]_q \alpha_6 \\
[12]_q \lambda^6_0(q) &= \{3\}_q \{4\}_q \alpha_1 + \{2\}_q [6]_q \alpha_2 + \{2\}_q \{3\}_q [4]_q \alpha_3 + [4]_q [6]_q \alpha_4 \\
&\quad + \{2\}_q \{3\}_q [5]_q \alpha_5 + \{3\}_q [8]_q \alpha_6
\end{align*}
\]

A.11. E7. In this case \( l = 18 \) and we have the following expressions:

\[
\begin{align*}
\{9\}_q \lambda^1_0(q) &= \{3\}_q [5]_q \alpha_1 + \{2\}_q \{3\}_q \alpha_2 + \{3\}_q [3]_q \alpha_3 + \{3\}_q [4]_q \alpha_4 \\
&\quad + [6]_q \alpha_5 + [2]_q \{3\}_q \alpha_6 + \{3\}_q \alpha_7 \\
\{9\}_q \lambda^2_0(q) &= \{2\}_q \{3\}_q \alpha_1 + \{3\}_q \{7\}_q / [2]_q \alpha_2 + \{3\}_q [4]_q \alpha_3 + \{2\}_q [6]_q \alpha_4 \\
&\quad + [3]_q [3]_q \alpha_5 + [6]_q \alpha_6 + \{3\}_q \alpha_7 \\
\{9\}_q \lambda^3_0(q) &= \{3\}_q [3]_q \alpha_1 + \{3\}_q [4]_q \alpha_2 + \{3\}_q [6]_q \alpha_3 + [2]_q \{3\}_q [4]_q \alpha_4 \\
&\quad + [2]_q [6]_q \alpha_5 + [2]_q^2 \{3\}_q \alpha_6 + [2]_q \{3\}_q \alpha_7 \\
\{9\}_q \lambda^4_0(q) &= \{3\}_q [4]_q \alpha_1 + \{2\}_q [6]_q \alpha_2 + \{2\}_q \{3\}_q [4]_q \alpha_3 + [4]_q [6]_q \alpha_4 \\
&\quad + [3]_q [6]_q \alpha_5 + [2]_q [6]_q \alpha_6 + [6]_q \alpha_7 \\
\{9\}_q \lambda^5_0(q) &= [6]_q \alpha_1 + [3]_q \{3\}_q \alpha_2 + [2]_q [6]_q \alpha_3 + [3]_q [6]_q \alpha_4 \\
&\quad + [3]_q [2]_q [5]_q \alpha_5 + [3]_q [5]_q \alpha_6 + [3]_q [5]_q \alpha_7 \\
\{9\}_q \lambda^6_0(q) &= \{2\}_q \{3\}_q \alpha_1 + [6]_q \alpha_2 + [2]_q^2 \{3\}_q \alpha_3 + [2]_q [6]_q \alpha_4 \\
&\quad + \{3\}_q [5]_q \alpha_5 + [2]_q \{3\}_q [4]_q \alpha_6 + \{3\}_q [4]_q \alpha_7 \\
\{9\}_q \lambda^7_0(q) &= \{3\}_q \alpha_1 + [3]_q \alpha_2 + [2]_q \{3\}_q \alpha_3 + [6]_q \alpha_4 \\
&\quad + [3]_q [5]_q \alpha_5 + [3]_q [4]_q \alpha_6 + [3]_q [4]_q \alpha_7
\end{align*}
\]
A.12. $E_8$. In this case $l = 30$ and we have the following expression:

$$
\{15\}_q \lambda_1^\alpha (q) = \{5\}_q [4]_q ^2 \alpha_1 + \{3\}_q [5]_q ^2 \alpha_2 + [2]_q \gamma \frac{\{5\}_q [7]_q}{[2]_q} \alpha_3 + \{3\}_q [10]_q \alpha_4 \\
+ \{3\}_q [4]_q \{5\}_q ^2 \alpha_5 + \{5\}_q [6]_q \alpha_6 + [2]_q \{3\}_q \{5\}_q ^2 \alpha_7 + \{3\}_q [5]_q \alpha_8 \\
\{15\}_q \lambda_2^\alpha (q) = \{3\}_q [5]_q ^2 \alpha_1 + \{3\}_q [5]_q [4]_q ^2 \alpha_2 + \{3\}_q [10]_q \alpha_3 + [3]_q [10]_q \alpha_4 \\
+ [2]_q \{5\}_q [6]_q \alpha_5 + \{3\}_q [3]_q [2]_q \{5\}_q ^2 \alpha_6 + \{5\}_q [6]_q \alpha_7 + \{5\}_q [3]_q [2]_q \alpha_8 \\
\{15\}_q \lambda_3^\alpha (q) = [2]_q \gamma ^{-1} \{5\}_q [7]_q \alpha_1 + \{3\}_q [10]_q \alpha_2 + [2]_q \gamma ^{-1} \{5\}_q [7]_q \alpha_3 + [2]_q \{3\}_q [10]_q \alpha_4 \\
+ [2]_q \{3\}_q [4]_q \{5\}_q ^2 \alpha_5 + [2]_q \{5\}_q [6]_q \alpha_6 + [2]_q ^2 \{3\}_q [5]_q \alpha_7 + [2]_q \{3\}_q [5]_q \alpha_8 \\
\{15\}_q \lambda_4^\alpha (q) = \{3\}_q [10]_q \alpha_1 + \{3\}_q [2]_q [10]_q \alpha_2 + [2]_q \{3\}_q [10]_q \alpha_3 + \{6\}_q [10]_q \alpha_4 \\
+ [4]_q \{5\}_q [6]_q \alpha_5 + \{3\}_q ^2 \{5\}_q [6]_q \alpha_6 + [2]_q \{5\}_q [6]_q \alpha_7 + \{5\}_q [6]_q \alpha_8 \\
\{15\}_q \lambda_5^\alpha (q) = \{3\}_q [4]_q [5]_q \alpha_1 + \{2\}_q [5]_q [6]_q \alpha_2 + [2]_q \{3\}_q [4]_q \{5\}_q \alpha_3 + \{4\}_q [5]_q [6]_q \alpha_4 \\
+ [2]_q \{3\}_q [10]_q \alpha_5 + \{3\}_q ^2 [10]_q \alpha_6 + \{3\}_q [10]_q \alpha_7 + \{3\}_q [5]_q \alpha_8 \\
\{15\}_q \lambda_6^\alpha (q) = \{5\}_q [6]_q \alpha_1 + \{3\}_q ^2 \{5\}_q \alpha_2 + [2]_q \{5\}_q [6]_q \alpha_3 + \{3\}_q [5]_q \{6\}_q \alpha_4 \\
+ \{3\}_q [10]_q ^2 \alpha_5 + \{4\}_q [5]_q \{6\}_q \alpha_6 + [2]_q ^2 \{3\}_q [4]_q \{5\}_q \alpha_7 + [2]_q ^2 \{4\}_q \{5\}_q \alpha_8 \\
\{15\}_q \lambda_7^\alpha (q) = [2]_q \{3\}_q ^2 \{5\}_q \alpha_1 + \{5\}_q [6]_q \alpha_2 + [2]_q \{3\}_q ^2 \{5\}_q \alpha_3 + \{2\}_q [5]_q \{6\}_q \alpha_4 \\
+ [3]_q [10]_q \alpha_5 + \{2\}_q [2]_q ^2 \{4\}_q \{5\}_q \alpha_6 + [2]_q [3]_q \{5\}_q ^2 \alpha_7 + [3]_q ^2 \{5\}_q \alpha_8 \\
\{15\}_q \lambda_8^\alpha (q) = \{3\}_q [5]_q \alpha_1 + \{5\}_q [3]_q [2]_q \alpha_2 + [2]_q \{3\}_q [5]_q \alpha_3 + \{5\}_q [6]_q \alpha_4 \\
+ \{3\}_q [5]_q ^2 \alpha_5 + [2]_q \gamma \{4\}_q \{5\}_q \alpha_6 + \{3\}_q ^2 \{5\}_q ^2 \alpha_7 + \{5\}_q \{9\}_q \alpha_8 \\

REFERENCES