Solutions to Practice Final

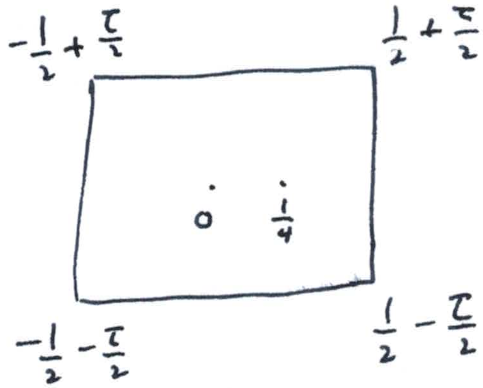
Problem 1

(1) Let $g(x,y) = x$. Then $g_x = 1 \quad g_y = 0$.

So Cauchy-Riemann equations do not hold and hence $f(z) = g(\text{Re}(z), \text{Im}(z))$ is not holomorphic

(2) $f(z) = z$ is holomorphic function on the entire complex plane. Since $g(w) = f\left(\frac{1}{w}\right) = \frac{1}{w}$ has order 1 pole at 0, $f(z)$ has a first order pole at infinity.

(3) There is no such function since within the parallelogram



Sum of zeroes $-$ Sum of poles

$$= \frac{1}{4} + \frac{1}{4} - 0 - 0 = \frac{1}{2}$$

is not of the form $m + n\tau$ for any $m, n \in \mathbb{Z}$.

Problem 2.

$$p(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

(1) Ratio of consecutive terms $= -z^2\left(\frac{2n+1}{2n+3}\right)$

$$\left| -z^2 \frac{2n+1}{2n+3} \right| \to |z|^2 \quad \text{as } n \to \infty.$$

So radius of convergence $= 1$.

(2) $p(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$ is convergent

by alternating series test : the partial sum from n^{th} term

to $(n+m)^{th}$ term this is sufficient.

$S_{n,m} = \frac{1}{2n+1} - \frac{1}{2n+3} \cdots (-1)^m \frac{1}{2n+1+2m}$ is positive (if m is

odd then it is $\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) + \left(\frac{1}{2n+5} - \frac{1}{2n+7}\right) + \cdots + \left(\frac{1}{2n-1+2m} - \frac{1}{2n+1+2m}\right)$

sum of positive term , and when m is even the sum in question

is $\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) + \cdots + \left(\frac{1}{2n-3+2m} - \frac{1}{2n-1+2m}\right) + \frac{1}{2n+1+2m}$ again a

sum of positive terms.)

Also $S_{n,m} < \frac{1}{2n+1}$ since $S_{n,m} = \frac{1}{2n+1} - S_{n+1,m-1} < \frac{1}{2n+1}$

again > 0.

Since $\lim_{n \to \infty} \frac{1}{2n+1} = 0$, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$ is convergent.

(given $\varepsilon > 0$, choose N such that $\frac{1}{2N+1} < \varepsilon$. then $|S_{n,m}| < \varepsilon$

for every $n \geq N$, $m \geq 0$. Hence the series is convergent by

Cauchy's criterion).

(3) $p'(z) = 1 - z^2 + z^4 - z^6 \cdots = \frac{1}{1+z^2}$ for $|z| < 1$

$$p(1) = \int_0^1 p'(x)\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_0^1 = \frac{\pi}{4}$$

Problem 3. (1) True. Proof Let $f(z)$ be a meromorphic function with infinitely many poles. For any $n = 1, 2, 3, \dots$, either $f(z)$ has finitely many poles in the disc $D_n(0)$ of radius n centered at 0, or there is some N such that $f(z)$ has infinitely many poles in $D_N(0)$.

In the second case, there must be a limit point of poles of $f(z)$ within $D_N(0)$, since it is closed and bounded. This limit point will be an essential singularity of $f(z)$.

In the first case, for every n, there is a pole a_n of $f(z)$ such that $|a_n| > n$. Then $\infty = \lim_{n \to \infty} a_n$ will be an essential singularity of $f(z)$.

(2) False. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is convergent by alternating series test, but

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$ is not. (as it is $> \int_1^\infty \frac{1}{x}\,dx = \infty$)

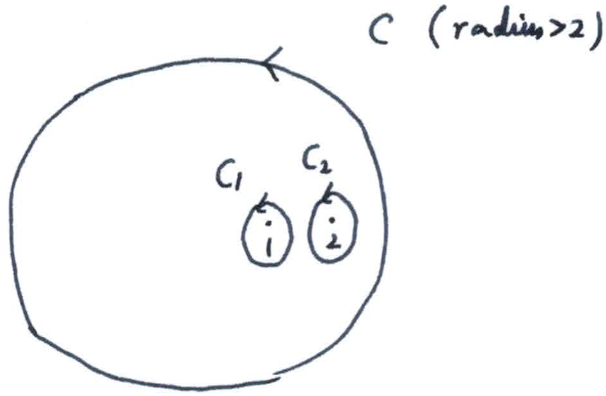
(3) False. $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{z^n}{n}$ has radius of convergence $= 1$ (given on the formula sheet)

for $w = 1$, $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent (by alternating series test)

Problem 4

$$\frac{z}{z^3 - 3z + 2} = \frac{z}{(z-1)^2 (z+2)}$$

$$\frac{1}{2\pi i} \int_C \frac{z}{(z-1)^2(z+2)} e^{zw} \, dz = \frac{1}{2\pi i} \int_{C_1} \frac{z}{(z-1)^2(z+2)} e^{zw} + \frac{1}{2\pi i} \int_{C_2} \frac{z}{(z-1)^2(z+2)} e^{zw} \, dz$$



by Cauchy's formula

$$= \left[\frac{d}{dz} \left(\frac{z e^{zw}}{z+2} \right) \right]_{\text{set } z=1} + \left[\frac{z}{(z-1)^2} e^{zw} \right]_{\text{set } z=-2}$$

$$= \left[\frac{(z+2)\left(e^{zw} + zw e^{zw}\right) - z e^{zw} \cdot 1}{(z+2)^2} \right]_{\text{set } z=1} + \frac{-2}{9} e^{-2w}$$

$$= \frac{3 e^{w} (1+w) - e^{w}}{9} - \frac{2}{9} e^{-2w} .$$

$$= \frac{1}{3} w \cdot e^{w} + \frac{2}{9} \left(e^{w} - e^{-2w} \right)$$

Problem 5.

$$\Gamma(z)\,\Gamma(1-z) = \pi \operatorname{cosec}(\pi z)$$

Set $z = \frac{1}{2}$ to get $\Gamma\left(\frac{1}{2}\right)^2 = \pi$.

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \left(\text{since } \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt > 0 \text{ for } x > 0 \text{ real}\right)$$

Problem 6. Set $t = \cos^2 x$. Then $dt = -2\cos(x)\sin(x)\,dx$

$$\sin(x) = (1-t)^{\frac{1}{2}} \qquad \cos(x) = t^{\frac{1}{2}}.$$

$$\int_0^{\pi/2} \cos^2(x)\sin^2(x)\,dx = -\frac{1}{2}\int_1^0 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt = \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{2}\right)^2}{\Gamma(3)}$$

(see the properties of gamma function on the formula sheet)

$$= \frac{1}{4}\Gamma\left(\tfrac{3}{2}\right)^2 \qquad \left(\Gamma(3) = (3-1)! = 2\right)$$

$$= \frac{1}{4}\left(\tfrac{1}{2}\right)^2 \Gamma\left(\tfrac{1}{2}\right)^2 \qquad \left(\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\right)$$

$$= \frac{1}{16}\Gamma\left(\tfrac{1}{2}\right)^2.$$

Problem 7. $$\theta(z) = \prod_{n\geq 1}(1-q^{2n}) \prod_{n\geq 0} 1-q^{2n} e^{2\pi i z} \prod_{n\geq 1} 1-q^{2n} e^{-2\pi i z}$$

$$= (1-e^{2\pi i z}) \prod_{n\geq 1} 1-q^{2n} \prod_{n\geq 1} 1-q^{2n} e^{2\pi i z} \prod_{n\geq 1} 1-q^{2n} e^{-2\pi i z}$$

$$\theta'(0) = -2\pi i \prod_{n\geq 1} (1-q^{2n})^3.$$

To compute $\theta'(2\tau)$, we used $\theta(z+\tau) = -e^{-2\pi i z}\theta(z)$.

$$\theta(z+2\tau) = -e^{-2\pi i(z+\tau)}\theta(z+\tau) = e^{-2\pi i\tau} e^{-4\pi i z}\theta(z)$$

$$\Rightarrow \theta'(z+2\tau) = e^{-2\pi i\tau}\left[e^{-4\pi i z}\theta'(z) - 4\pi i e^{-4\pi i z}\theta(z)\right]$$

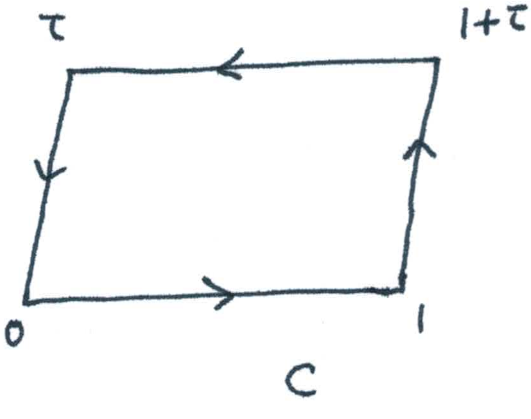
Set $z=0$: $\theta'(2\tau) = e^{-2\pi i\tau}\theta'(0)$
$(\theta(0)=0)$

$$= e^{-2\pi i\tau}(-2\pi i)\prod_{n\geq 1}(1-q^{2n})^3$$

Problem 8. $f(z+1) = f(z)$ $\quad f(z+\tau) = f(z) - 2\pi i$

Sum of residues of $f(z)$ at poles within C

$$= \frac{1}{2\pi i}\int_C f(z)\,dz$$



$$= \frac{1}{2\pi i}\left[\int_0^1 f(z)\,dz + \int_1^{1+\tau} f(z)\,dz - \int_0^{\tau} f(z)\,dz - \int_{\tau}^{1+\tau} f(z)\,dz \right]$$

$$= \frac{1}{2\pi i}\left[\int_0^1 f(z)\,dz + \int_0^{\tau} f(w+1)\,dw - \int_0^{\tau} f(z)\,dz - \int_0^1 f(w+\tau)\,dw \right]$$

$$= \frac{1}{2\pi i}\left[\cancel{\int_0^1 f(z)\,dz} + \cancel{\int_0^{\tau} f(w)\,dw} - \cancel{\int_0^{\tau} f(z)\,dz} - \int_0^1 (\cancel{f(w)} - 2\pi i)\,dw \right]$$

$$= \frac{1}{2\pi i} \cdot 2\pi i = 1.$$

<u>Problem 9.</u> Periodicity properties of theta functions:

$$\theta_1(z+\tau) = -e^{-2\pi i z}\,\theta_1(z)$$
$$\theta_3(z+\tau) = q^{-1} e^{-2\pi i z}\,\theta_3(z)$$
$$\theta_4(z+\tau) = -q^{-1} e^{-2\pi i z}\,\theta_4(z)$$

Let $f(z) = e^{2\pi i z}\, \dfrac{\theta_4(0)^2\theta_3(z)^2 - \theta_3(0)^2\theta_4(z)^2}{\theta_1(z)^2}$

then $f(z+1) = f(z)$ clearly.

$$f(z+\tau) = q^2 e^{2\pi i z}\left[q^{-2} e^{-4\pi i z} \frac{\left(\theta_4(0)^2\theta_3(z)^2 - \theta_3(0)^2\theta_4(z)^2\right)}{e^{-4\pi i z}\ \theta_1(z)^2}\right]$$ ⑧

$$= f(z).$$

$\Rightarrow$ $f(z)$ is elliptic with possibly one pole of order 2 at $z=0$ with a fundamental parallelogram. But the numerator at $z=0$ is also 0. So this pole is at most of order 1. Therefore it is a removable pole and hence $f(z)$ is holomorphic. Since every holomorphic and elliptic function has to be a constant

$$f(z) = \text{constant } C. \qquad C = f\left(\tfrac{\tau}{2}\right) \quad \left(\begin{array}{l}\text{Note} \\ \theta_4(\tfrac{\tau}{2}) = 0\end{array}\right)$$

$$C = f\left(\tfrac{\tau}{2}\right) = e^{2\pi i \frac{\tau}{2}} \frac{\cancel{\theta_4(0)^2}\,\theta_3(\tau/2)^2}{\cancel{\theta_1(\tau/2)^2}} \qquad \left(\theta_4(0) = \theta_1(0+\tfrac{\tau}{2})\right)$$

$$= q\ \theta_3\left(\tfrac{\tau}{2}\right)^2 = q\ \theta_1\left(\tfrac{\tau}{2}+\tfrac{1}{2}+\tfrac{\tau}{2}\right)^2$$

$$= q\ \theta_1\left(\tfrac{1}{2}+\tau\right)^2 = q\left(-e^{-2\pi i(\frac{1}{2})}\right)^2 \theta_1\left(\tfrac{1}{2}\right)^2$$

$$= q\ \theta_1\left(\tfrac{1}{2}\right)^2.$$

Hence we get

$$e^{2\pi i z}\left(\theta_4(0)^2\theta_3(z)^2 - \theta_3(0)^2\theta_4(z)^2\right) = q\ \theta_1\left(\tfrac{1}{2}\right)^2\theta_1(z)^2.$$