COMPLEX VARIABLES: PRACTICE FINAL

Problem 1. Give example or prove that there are none.

(1) A function of two real variables $g(x, y)$ such that all partial derivatives of all orders exist, but $f(z) = g(\mathrm{Re}(z), \mathrm{Im}(z))$ is not holomorphic.

(2) A holomorphic function on the entire complex plane with a first order pole at infinity.

(3) An elliptic function (i.e, a meromorphic function $f(z)$ such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$ where $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$) such that within the parallelogram with vertices $\pm\frac{1}{2} \pm \frac{\tau}{2}$ the function has only one second order pole at 0 and only one zero of order two at $\frac{1}{4}$.

Problem 2. Consider the following power series:

$$p(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

(1) What is the radius of convergence of $p(z)$?

(2) Prove that $p(1)$ is a convergent series.

(3) Compute $p'(z)$ and use it to determine the value of $p(1)$.

Problem 3. True or False. Justify your answer with a proof or a counterexample.

(1) Every meromorphic function with infinitely many poles must have an essential singularity, either at a complex number or at infinity.

(2) For complex numbers $c_1, c_2, \cdots$, if $\sum_{n=0}^{\infty} c_n$ is convergent, then so must be $\sum_{n=0}^{\infty} |c_n|$.

(3) If a power series $\sum_{n=0}^{\infty} a_n z^n$ has a finite non–zero radius of convergence, say r, then for every $w \in \mathbb{C}$ with $|w| = r$, the series $\sum_{n=0}^{\infty} a_n w^n$ is divergent.

Problem 4. Compute the following integral where C is the counterclockwise circle of radius $r > 2$ centered at 0.

$$\frac{1}{2\pi i} \int_C \frac{z}{z^3 - 3z + 2} e^{zw} \, dz$$

Problem 5. Compute the value of $\Gamma\left(\frac{1}{2}\right)$.

Problem 6. Prove that

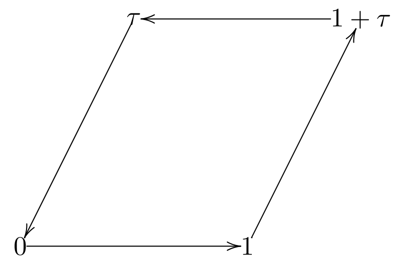
$$\int_0^{\frac{\pi}{2}} \cos^2(x)\sin^2(x)\,dx = \frac{1}{16}\Gamma\left(\frac{1}{2}\right)^2$$

Problem 7. Find the value of $\theta'(0)$. What is the value of $\theta'(2\tau)$?

Problem 8. Let $f(z)$ be a meromorphic function satisfying

$$f(z+1) = f(z) \qquad\qquad f(z+\tau) = f(z) - 2\pi i$$

Assume $f(z)$ has no poles on the following simple closed path C:



Prove that the sum of residues of $f(z)$ at poles lying within C is 1.

Problem 9. Prove the following identity between the theta functions (see the last page for definitions).

$$e^{2\pi iz}\left(\theta_4(0)^2\theta_3(z)^2 - \theta_3(0)^2\theta_4(z)^2\right) = q\theta_1\left(\frac{1}{2}\right)^2\theta_1(z)^2$$

List of useful results.

A few power series expansions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for any } z$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1$$

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \text{ for } |z| < 1$$

Alternating series test: if $b_1 \geq b_2 \geq \cdots \geq 0$ are real numbers such that $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n\geq 0} (-1)^n b_n$ is convergent.

Properties of the gamma function

$$\Gamma(z)^{-1} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

(1) $\Gamma(z+1) = z\Gamma(z)$.
(2) $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$.
(3) Relation with Eulerian integrals: for any a, b with positive real part,

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$. Let $q = e^{\pi i \tau}$.

Properties of an elliptic function $f(z)$.

(1) If holomorphic then constant.
(2) Sum of residues at poles within a fundamental parallelogram is zero.
(3) Number of zeroes = Number of poles within a fundamental parallelogram.
(4) Sum of zeroes = sum of poles + $m + n\tau$ for some $m, n \in \mathbb{Z}$. (zereos/poles within a fundamental parallelogram).

Properties of the theta function.

$$\theta(z;\tau) = \theta_1(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z} = \prod_{n\geq 1}(1-q^{2n}) \prod_{n\geq 0}(1-q^{2n}e^{2\pi i z}) \prod_{n\geq 1}(1-q^{2n}e^{-2\pi i z})$$

(1) $\theta(z+1) = \theta(z)$ and $\theta(z+\tau) = -e^{-2\pi i z}\theta(z)$.
(2) $\theta(z) = 0$ if and only if $z = m + n\tau$ $(m, n \in \mathbb{Z})$ each with multiplicity 1.
(3) $\theta_2(z) = \theta(z + \frac{1}{2})$. $\theta_3(z) = \theta(z + \frac{1+\tau}{2})$. $\theta_4(z) = \theta(z + \frac{\tau}{2})$.