Solutions to Practice Mid Term I

(1) $$f(z) = e^{x}\left(\cos(y) + i\sin(y)\right) \quad \text{where} \quad x = \mathrm{Re}(z),\ y = \mathrm{Im}(z)$$

$$u(x,y) = e^{x}\cos(y) \qquad v(x,y) = e^{x}\sin(y)$$ (real and imaginary parts of $f(z)$)

$$u_x = e^{x}\cos(y) \qquad v_x = e^{x}\sin(y)$$

$$u_y = -e^{x}\sin(y) \qquad v_y = e^{x}\cos(y)$$

Hence, $u_x = v_y$ and $u_y = -v_x$ (Cauchy-Riemann equations).

By Cauchy-Riemann Theorem $f(z)$ is $\mathbb{C}$-differentiable and

$$f'(z) = u_x + i v_x = e^{x}\cos(y) + i e^{x}\sin(y) = f(z).$$

(2) $$p(z) = c_n z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0 \ ; \quad c_0, c_1, \dots, c_n \in \mathbb{R}$$

$z_0 \in \mathbb{C}$ is a solution of $p(z) = 0$, that is,

$$c_n z_0^n + c_{n-1} z_0^{n-1} + \dots + c_1 z_0 + c_0 = 0$$

Take conjugate of this equation and use $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\cdot\overline{z_2}$

and $\overline{c_k} = c_k$ for every $k = 0, 1, 2, \dots, n$ since $c_k \in \mathbb{R}$.

$$c_0 (\overline{z_0})^n + c_{n-1} (\overline{z_0})^{n-1} + \dots + c_1 (\overline{z_0}) + c_0 = 0$$

Hence $\overline{z_0}$ is also a solution of $p(z) = 0$.

By fundamental theorem of algebra, $p(z)$ can be written as

$$p(z) = c_n (z - a_1)(z - a_2) \dots (z - a_n)$$

$a_1, a_2, \dots, a_n \in \mathbb{C}$.

Now every root of $p(z)$ is either real, or comes as a pair $(a, \bar{a})$.

$$(z-a)(z-\bar{a}) = z^2 - (a+\bar{a})z + a\bar{a}$$

$$= z^2 - 2\,\mathrm{Re}(a)\,z + |a|^2$$ has only real coefficients.

Therefore $p(z)$ can be written as a product of degree 1 and degree 2 polynomials with real coefficients.

(3) $u(x,y) = xy$. If $f(z) = u(x,y) + i\,v(x,y)$ is $\mathbb{C}$-diff.

then $v_y = u_x = y$ by Cauchy-Riemann equations.

$v_x = -u_y = -x$

$v_y = y \Rightarrow v = \frac{y^2}{2} + g(x)$ for some function $g(x)$

$\Rightarrow -x = v_x = g'(x) \Rightarrow g(x) = -\frac{x^2}{2} + \text{constant}.$

Thus $f(z) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$ is a $\mathbb{C}$-diff. function with real part $x \cdot y$.

(4) $$\frac{2z-1}{(z-3)(z-1-i)^2} = \frac{a}{z-3} + \frac{b_1}{z-1-i} + \frac{b_2}{(z-1-i)^2}$$

Multiply both sides by $(z-1-i)$:

$$\frac{2z-1}{(z-3)(z-1-i)} = \frac{a(z-1-i)}{z-3} + b_1 + \frac{b_2}{z-1-i}$$

Let C be a simple closed curve such that $1+i \in \mathrm{Int}(C)$ and 3 is not in the interior or on C.

(e.g. C = counterclockwise circle of radius $r < |3-(1+i)| = \sqrt{5}$ centered at $1+i$)

$$\int_C \frac{2z-1}{z-3}\left(\frac{1}{z-1-i}\right)dz = 2\pi i \, \frac{2(1+i)-1}{1+i-3}$$ by Cauchy's integral formula.

and $$\int_C \frac{a(z-1-i)}{z-3}\,dz = 0 \qquad \int_C b_1\,dz = 0$$

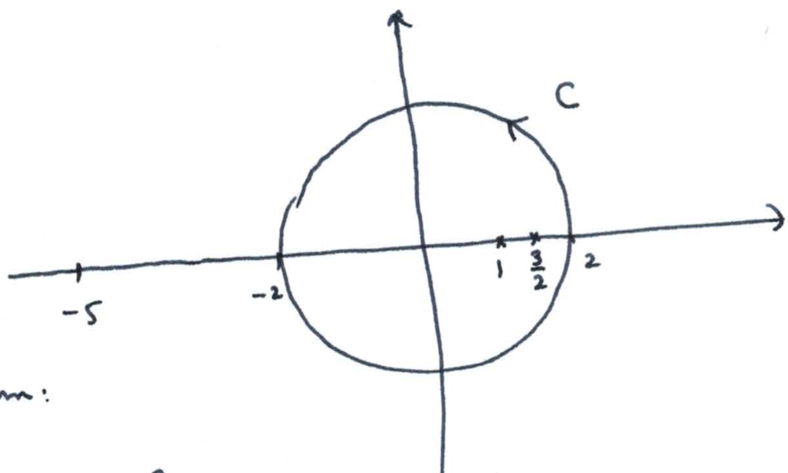
$$\int_C \frac{b_2}{z-1-i}\,dz = 2\pi i\, b_2$$

Hence $$2\pi i\, b_2 = 2\pi i\,\frac{2(1+i)-1}{1+i-3} \Rightarrow b_2 = \frac{1+2i}{-2+i}.$$

(5) $$f(z) = \frac{z^2-2}{(z+5)(2z-3)(z-1)} = \frac{z^2-2}{2(z+5)(z-\frac{3}{2})(z-1)}$$ is $\mathbb{C}$-diff. everywhere except $-5, \frac{3}{2}$ and 1.

C = circle of radius 2 centred at 0

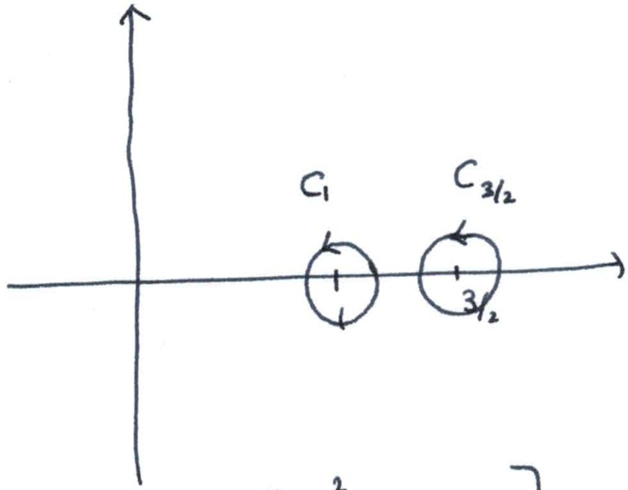
$1, \frac{3}{2} \in \text{Int}(C)$ and $-5 \notin \text{Int}(C)$



Application of Cauchy's Theorem:

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_{3/2}} f(z)\,dz$$ where

C_1 and $C_{\frac{3}{2}}$ are circles ~~are~~ centered at 1 and $\frac{3}{2}$ resp. of radii $< \frac{1}{4}$.



$$\int_{C_1} f(z)\,dz = \int_{C_1} \frac{z^2-2}{(z+5)(2z-3)}\left(\frac{1}{z-1}\right)dz = 2\pi i\left[\frac{1^2-2}{(1+5)(2(1)-3)}\right]$$

$$= 2\pi i\left(\frac{1}{6}\right)$$

and

$$\int_{C_{3/2}} f(z)\,dz = \int_{C_{3/2}} \frac{z^2-2}{2(z+5)(z-1)}\left(\frac{1}{z-\frac{3}{2}}\right)dz = 2\pi i\left[\frac{(3/2)^2-2}{2\left(\frac{3}{2}+5\right)\left(\frac{3}{2}-1\right)}\right]$$

$$= 2\pi i\left(\frac{0.25}{6.5}\right) = 2\pi i\left(\frac{1}{4}\;\frac{2}{13}\right) = 2\pi i\left(\frac{1}{26}\right)$$

Hence

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{6}+\frac{1}{26}\right) = 2\pi i\;\frac{13+3}{78}$$

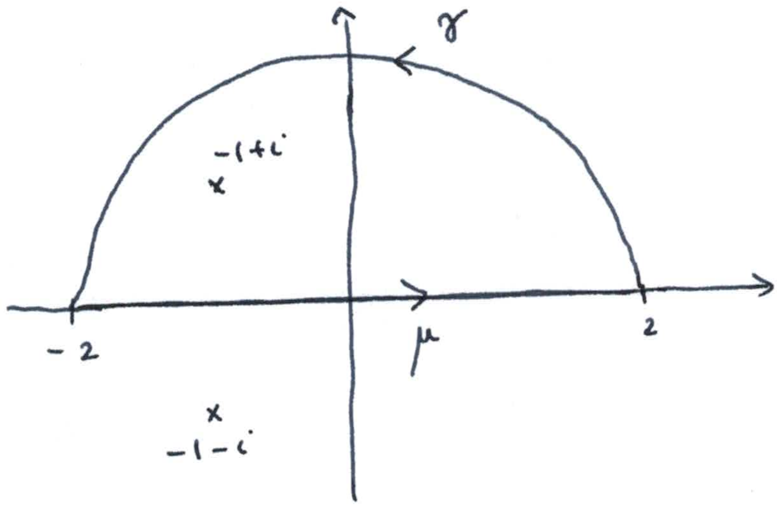
$$= \frac{16}{78}\cdot 2\pi i$$

□

(6) $z^2 + 2z + 2 = 0$ implies $z = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$= \dfrac{-2 \pm 2i}{2} = -1 \pm i$

Therefore $\dfrac{1}{z^2+2z+2}$ is $\mathbb{C}$-diff. everywhere except $-1+i$ and $-1-i$



$\gamma(t) = 2(\cos(t) + i\sin(t))$
$0 \leq t \leq \pi$

Let μ be the straight line joining -2 and 2

$\mu(t) = t \quad (-2 \leq t \leq 2)$

By Cauchy's integral formula

$$\int_{\gamma+\mu} \frac{1}{z^2+2z+2}\,dz = \int_{\gamma+\mu} \frac{1}{z-(-1-i)} \frac{1}{z-(-1+i)}\,dz = 2\pi i \left(\frac{1}{-1+i+1+i}\right)$$

$$= 2\pi i \frac{1}{2i} = \pi.$$

$$\text{Now } \int_{\mu} \frac{1}{z^2+2z+2}\,dz = \int_{-2}^{2} \frac{1}{t^2+2t+2}\,dt = \int_{-2}^{2} \frac{1}{1+(t+1)^2}\,dt$$

$$= \left[\tan^{-1}(t+1)\right]_{-2}^{2} = \tan^{-1}(3) - \tan^{-1}(-1)$$

Hence $$\int_\gamma \frac{1}{z^2+2z+2}\,dz = \pi - \int_\mu \frac{1}{z^2+2z+2}\,dz$$

$$= \pi - \tan^{-1}(3) + \tan^{-1}(-1)$$

(7) $$f(z) = \frac{z+1}{z-1}$$

If $z = it$ $(t \in \mathbb{R})$ then $f(it) = \dfrac{it+1}{it-1} = -\left(\dfrac{1+it}{1-it}\right)$

$|1+it| = \sqrt{1+t^2} = |1-it|$ implies for every $t \in \mathbb{R}$

$|f(it)| = 1$. Hence values of $f(it)$ always lie on

the circle of radius 1 centered at 0.