PRACTICE MID TERM I: COMPLEX VARIABLES

You are allowed to use the results written on the formula sheet (last page). Write complete solutions (not just the answer) for full credit.

(1) Prove that the following function is $\mathbb{C}$–differentiable.
$$f(z) = e^{\mathrm{Re}(z)} \left(\cos\left(\mathrm{Im}(z)\right) + i \sin\left(\mathrm{Im}(z)\right)\right)$$
Prove that $f'(z) = f(z)$.

(2) Let $p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ be a polynomial of degree n, with real coefficients. That is, $c_0, c_1, \cdots, c_n \in \mathbb{R}$. Prove that if $z_0 \in \mathbb{C}$ is a solution of $p(z) = 0$, then so is $\overline{z_0}$. Using this, prove that $p(z)$ can be written as a product of degree 1 and degree 2 real polynomials.

(3) Let $u(x, y) = xy$. Find a $\mathbb{C}$–differentiable function $f(z)$ whose real part is $u(x, y)$.

(4) Consider the partial fraction decompositon:
$$\frac{2z-1}{(z-3)(z-1-i)^2} = \frac{a}{z-3} + \frac{b_1}{z-1-i} + \frac{b_2}{(z-1-i)^2}$$
Compute the coefficient b_2 in the decomposition above.

(5) Let C be (counterclockwise oriented) circle of radius 2 centered at 0. Compute
$$\int_C \frac{z^2-2}{(z+5)(2z-3)(z-1)}\, dz$$

(6) Let $\gamma(t) = 2(\cos(t) + i\sin(t))$, where $0 \le t \le \pi$. Compute
$$\int_\gamma \frac{1}{z^2+2z+2}\, dz$$

(7) Consider the function $f(z) = \dfrac{z+1}{z-1}$. Prove that for z lying on the vertical line $\mathrm{Re}(z) = 0$ (this is just the imaginary axis, i.e, $z = it$ for $t \in \mathbb{R}$), the values of the function f lie on a circle. Find the radius and center of this circle.
(**Note.** The same is true for any vertical line, except for $\mathrm{Re}(z) = 1$. You can practice proving it, say, for example, for the vertical line $\mathrm{Re}(z) = 2$.)

Formula sheet

Statements of important results.

(1) $f(z) = u(x, y) + iv(x, y)$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Then f is $\mathbb{C}$–differentiable if, and only if the partial derivatives of u and v exist and are continuous, and $u_x = v_y$ and $u_y = -v_x$.

(2) **Cauchy's Theorem.** If $f : D \to \mathbb{C}$ is $\mathbb{C}$–differentiable, where $D \subset \mathbb{C}$ is a non–empty open set, and γ is a simple closed curve whose interior is contained in D, then $\int_\gamma f(z)\, dz = 0$.

(3) **Cauchy's Integral formula.** With the same hypotheses on f, D, γ as in the previous statement, assume $a \in \mathrm{Int}(\gamma)$, then $\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$.

(4) Let $p(z), q(z)$ be two polynomials of degree n and m respectively. Let γ be a simple closed curve, which contains all the zeroes of $q(z)$ in its interior. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{p(z)}{q(z)}\, dz = \begin{cases} 0 & \text{if} \quad m \geq n+2 \\ \frac{a_n}{b_{n+1}} & \text{if} \quad m = n+1 \end{cases}$$

where a_n, b_m are coefficients of z^n, z^m in $p(z)$ and $q(z)$ respectively.

Anti-derivatives of a few functions.

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}\ (n \neq -1) \qquad \int e^x\, dx = e^x$$

$$\int \sin(x)\, dx = -\cos(x) \qquad \int \cos(x)\, dx = \sin(x)$$

$$\int \sec(x)\tan(x)\, dx = \sec(x) \qquad \int \sec^2(x)\, dx = \tan(x)$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}(x) \qquad \int \frac{1}{1+x^2}\, dx = \tan^{-1}(x)$$

Trigonometric identities.

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sec^2(x) - \tan^2(x) = 1$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$