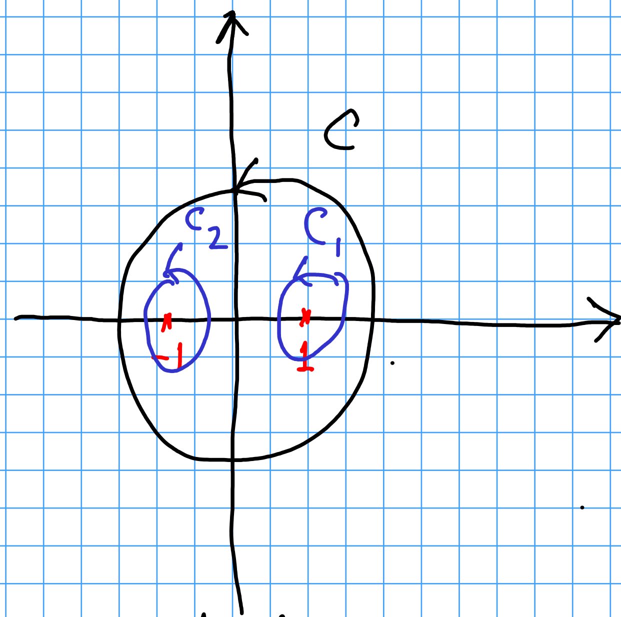
Solutions to Mid Term I

Problem 1. $\displaystyle\int_C \frac{3z^3+2}{z^2-1}\,dz$



counterclockwise circle of radius 2.

Solution I.

$$\int_C \frac{3z^3+2}{(z-1)(z+1)}\,dz$$

$$= \int_{C_1} \frac{3z^3+2}{(z-1)(z+1)}\,dz + \int_{C_2} \frac{3z^3+2}{(z-1)(z+1)}\,dz \quad \text{(by Cauchy's Thm)}$$

$$= 2\pi i \left[\frac{5}{2} + \frac{-1}{-2} \right] = 6\pi i \quad \text{(by Cauchy's integral formula)}$$

Solution II.

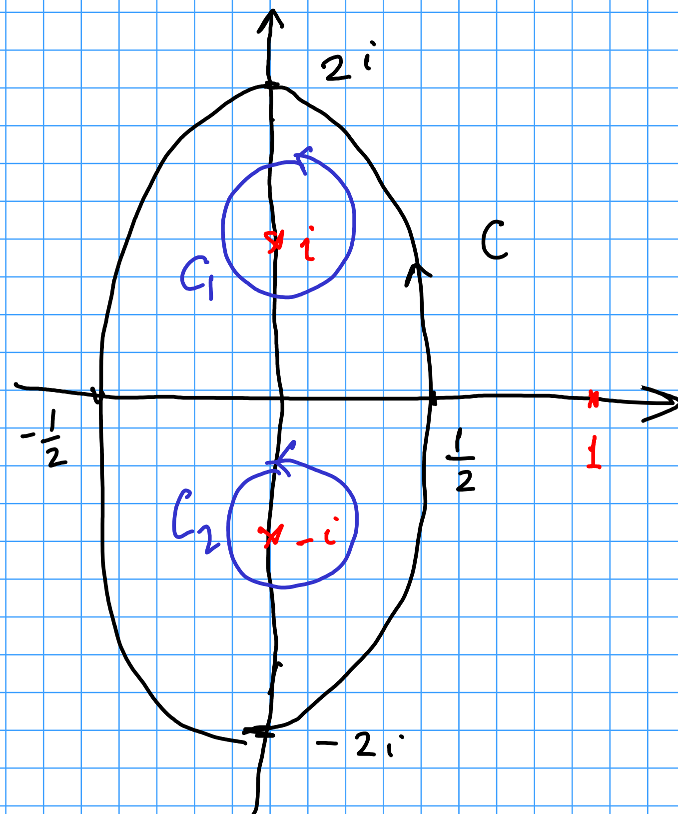
$$\frac{3z^3+2}{z^2-1} = 3z + \frac{3z+2}{z^2-1}$$

$$\int_C 3z\,dz = 0 \quad \text{and} \quad \int_C \frac{3z+2}{z^2-1}\,dz = 2\pi i\left(\frac{3}{1}\right)$$

by (4) of Formula sheet.

Problem 2.

$$\int_C \frac{1}{z^3 - z^2 + z - 1}\, dz$$



$$4x^2 + \frac{y^2}{4} = 1$$

$$z^3 - z^2 + z - 1$$
$$= z^2(z-1) + 1(z-1)$$
$$= (z-1)(z^2+1)$$
$$= (z-1)(z-i)(z+i)$$

By Cauchy's Theorem

$$\int_C \frac{1}{(z-1)(z-i)(z+i)}\, dz$$

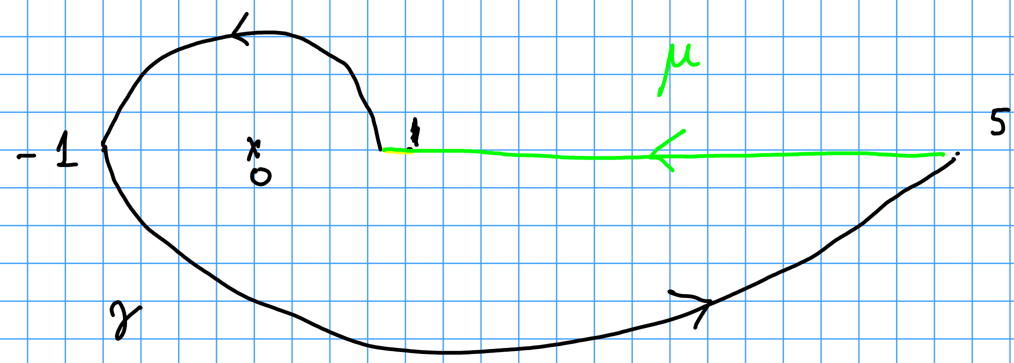
$$= \int_{C_1} \frac{1}{(z-1)(z-i)(z+i)}\, dz + \int_{C_2} \frac{1}{(z-1)(z-i)(z+i)}\, dz$$

$$= 2\pi i \left[\frac{1}{(i-1)\,2i} + \frac{1}{(-i-1)(-2i)} \right] = \pi \left[\frac{1}{i-1} + \frac{1}{i+1} \right]$$

$$= \pi \frac{2i}{-2} = -\pi i$$

Problem 3.

$$\gamma(t) = \begin{cases} \cos(t) + i\sin(t) & 0 \le t \le \pi \\ 2 + 3\left(\cos(t) + i\sin(t)\right) & \pi \le t \le 2\pi \end{cases}$$



By Cauchy's integral formula $\displaystyle\int_{\gamma+\mu} \frac{1}{z}\,dz = 2\pi i$

$$\Rightarrow \int_{\gamma} \frac{1}{z}\,dz = 2\pi i - \int_{\mu} \frac{1}{z}\,dz$$

$$= 2\pi i + \int_1^5 \frac{1}{t}\,dt = 2\pi i + \left[\ln(t)\right]_1^5$$

$$= 2\pi i + \ln(5) - \ln(1)$$

$$= 2\pi i + \ln(5).$$

Problem 4.

$$f(z) = (e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin(y)$$

$$u(x,y) = (e^x + e^{-x})\cos(y) \quad \text{and} \quad v(x,y) = (e^x - e^{-x})\sin y$$

$$\Rightarrow \quad u_x = (e^x - e^{-x})\cos(y) \qquad v_x = (e^x + e^{-x})\sin(y)$$

$$u_y = -(e^x + e^{-x})\sin(y) \qquad v_y = (e^x - e^{-x})\cos(y)$$

Hence $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ Cauchy-Riemann eq^{n}s

implying that f is $\mathbb{C}$-diff. Moreover

$$f'(z) = u_x + i v_x$$

$$= (e^x - e^{-x})\cos(y) + i(e^x + e^{-x})\sin(y)$$

which is again $\mathbb{C}$-diff. Since

$$\frac{\partial}{\partial x}\left((e^x - e^{-x})\cos(y)\right) = (e^x + e^{-x})\cos(y)$$

$$= \frac{\partial}{\partial y}\left((e^x + e^{-x})\sin(y)\right)$$

$$\frac{\partial}{\partial y}\left((e^x - e^{-x})\cos(y)\right) = -(e^x - e^{-x})\sin(y)$$

$$= -\frac{\partial}{\partial x}\left((e^x + e^{-x})\sin(y)\right)$$

and $$f''(z) = (e^x + e^{-x})\cos(y) + i(e^x - e^{-x})\sin(y) = f(z).$$

Problem 5. $f(z) = \dfrac{1}{z-1}$ and $z = 2+it$ $(t \in \mathbb{R})$

$$f(2+it) = \frac{1}{2+it-1} = \frac{1}{1+it}$$

$$= \frac{1}{1+it} \frac{1-it}{1-it} = \frac{1-it}{1+t^2}$$

So $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{-t}{1+t^2}$.

Set $t = \tan(\theta)$. Then $x = \dfrac{1}{1+\tan^2\theta} = \cos^2\theta$

$$= \frac{1}{2}\left(1+\cos(2\theta)\right)$$

$$y = \frac{-t}{1+t^2} = \frac{-\tan\theta}{\sec^2\theta} = -\sin\theta\cos\theta = \frac{-1}{2}\sin(2\theta).$$

Since $\cos^2(2\theta) + \sin^2(2\theta) = 1$, we get

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

Hence the values of f for $z = 2+it$ always lie on the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$.